BNL-HET-02/16
IJS/TP-28/02
Rome-1340/02

$B \to \pi$ and $B \to K$ transitions in standard and quenched chiral perturbation theory

Damir Bećirević[a], Saša Prelovšek[b,c,d] and Jure Zupan[c]

[a]*Dip. di Fisica, Università di Roma "La Sapienza", Piazzale Aldo Moro 2, I-00185 Rome, Italy.*

[b]*Theory Group, Brookhaven National Laboratory, Upton, NY 11973, USA.*

[c]*J. Stefan Institute, Jamova 39, P.O. Box 3000, 1001 Ljubljana, Slovenia.*

[d]*Departement of Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia.*

Abstract

We study the effects of chiral logs on the *heavy* $\to$ *light* pseudoscalar meson transition form factors by using the standard and quenched chiral perturbation theory combined with the static heavy quark limit. Resulting expressions are used to indicate the amount of uncertainties due to the use of the quenched approximation in the current lattice studies. They may also be used to assess the amount of systematic uncertainties induced by missing chiral log terms in extrapolating towards the physical pion mass. We also provide the coefficient multiplying the quenched chiral log, which may be useful if the quenched lattice studies are performed with very light mesons.

PACS:12.39.Fe (Chiral Lagrangians), 12.39.Hg (Heavy quark effective theory), 13.20.-v (Leptonic and semileptonic decays of mesons), 11.15.Ha (Lattice gauge theory).

1 Introduction

Over the past decade, a considerable effort has been put in studying the non-perturbative QCD dynamics of the *heavy* $\to$ *light* decays. The main target was (and still is) the extraction of the CKM matrix element $|V_{ub}|$. The prerequisite for its determination from the exclusive $B \to \pi\ell\nu$ decay mode is the precise knowledge of the relevant form factors. The accurate information on the form factors is crucial also when studying the impact of the physics beyond the Standard Model (SM) on the exclusive $b \to s\ell^+\ell^-$ modes.

The fact that the kinematically accessible region of momentum transfers is very large (e.g. for $B \to \pi\ell\nu$ it is $0 \le q^2 \le 26.4\ \mathrm{GeV}^2$) makes the QCD-based calculations of form factors ever more difficult. The physical pictures emerging on the two extremities of the q^2-region are quite different and effective field theory approaches, based on the appropriate use of the heavy quark expansion, have been developed to simplify the description of these processes. The heavy quark effective theory (HQET), which is applicable for the recoil momenta close to zero ($q^2 \to q^2_{\rm max}$) [1], provides us with valuable scaling laws for the form factors [1]. In the region of large recoils ($q^2 \to 0$), instead, the large energy effective theory and its descendants help resolving the heavy quark dependence of the form factors [2, 3]. Although these conceptual steps forward are highly beneficial for a better understanding of the underlying dynamics, a model-independent description (calculation) of the form factors in the entire physical region is still missing.

Among the QCD-based approaches employed to compute the *heavy* $\to$ *light* decay form factors, the following two stand apart:

- ⊗ Light cone QCD sum rules (LCSR). This analytic approach contains the least number of assumptions and has the correct heavy quark mass scaling properties. The range of applicability is however limited to low q^2's [4].

- ⊗ Lattice QCD. This method allows us to solve the non-perturbative QCD effects numerically. Because of the current insufficient computing power, the $B \to \pi$ transition is reached either: (a) by extrapolating the directly computed form factors from the region around the charm to the b-quark mass [5, 6], or (b) by using a latticised effective theory of heavy quark, such as NRQCD [7] (see also ref. [8]), or (c) by reinterpreting the lattice QCD in terms of NRQCD when the heavy quark mass becomes larger than the lattice UV cut-off [9]. All these strategies share one feature: the accessible form factors are restricted to the region of small recoils.

It is fair to say that the LCSR and lattice QCD are complementary to each other; it is important to use them both in order to check their consistency and from their comparison perhaps learn more about the underlying non-perturbative QCD dynamics.

Since the lattice studies are expected to provide us with the most accurate predictions about the shapes and absolute values of the form factors, it is important to have a good control over various assumptions that are currently used in the lattice simulation and the data analysis. Two sources of systematic uncertainty have so far been ignored: the

[1] Zero recoil refers to the recoil of the daughter meson in the rest frame of the decaying one. In $B \to \pi\ell\nu$, it means that the pion is soft.

quenched approximation and possible deviations from the linear/quadratic chiral extrapolation forms.

All the available lattice results for $B \to \pi\ell\nu$ decay form factors are obtained from simulations in the quenched approximation, where the sea quark loop effects in the QCD vacuum fluctuations are neglected ($n_F = 0$). To get an idea about the systematic error induced by the quenched approximation, one can confront the expressions for the form factors derived in the standard and in the quenched chiral perturbation theory (ChPT and QChPT, respectively). Such expressions, at leading order in the heavy quark expansion and the next-to-leading order (NLO) in the chiral expansion, are provided in the present paper. These expressions are also useful in assessing the systematic uncertainties due to chiral extrapolations. Current lattice studies deal with light mesons of masses $\gtrsim 450$ MeV. The physical pion mass is reached through a linear or quadratic extrapolation in the light quark mass. Although it is not clear for which light quark masses one begins to probe the subtleties of the chiral expansion, it is beyond a reasonable doubt that, very close to the chiral limit, the chiral log terms of the form $m_\pi^2 \log(m_\pi^2)$ may modify the result of the extrapolation. The coefficients multiplying the chiral logs are predicted by (Q)ChPT and will be presented in this paper. Finally, in the case of $B \to K$ decay the standard lattice strategy is to consider kaon as a meson consisting of degenerate quark masses. The impact of non-degeneracy in the quenched approximation may also be addressed by using the QChPT expressions for the form factors, as we shall see later on.

It is important to stress a difficulty in getting reliable numerical estimates from this approach. As we just mentioned, it is not clear for what value of the light meson mass the chiral logs become relevant. That ambiguity is important since one extrapolates from the heavier light masses, for which the chiral logs have *not* been observed. Another obstacle is a multitude of low energy constants that appear in the Lagrangian and in the transition operators in both the quenched and unquenched ChPT. The values of some of those constants are unknown or simply guessed. Furthermore, as we shall see, the appearance of the chirally divergent quenched chiral logs obliges us to compare the full and quenched expressions for not-so-light mesons, for which the ChPT is less predictive. For these three reasons, the numerical results inferred from this approach should always be taken with a grain of salt. In other words, rather than the true estimates of the quenching errors, our numerical results should be considered as a mere indicator of the size of those errors.

The rest of the paper is organized as follows: in section 2 we recall the basic elements of the (Q)ChPT and its combination with the leading order HQET Lagrangian; besides standard definitions of the form factors, we will introduce the ones that are more convenient for our purposes; in section 3 we present the one-loop (standard and quenched) chiral corrections for our form factors; in section 4 we discuss the values of the low energy constants that we chose for the numerical analysis, which we present in sections 5 and 6; we conclude in section 7.

2 Setting up the scene

In this section we recall some basic features of the QChPT, as it has been developed in the papers by Sharpe [10] and by Bernard and Golterman [11] [2]. Although the QChPT resembles the standard ChPT in many aspects there are important differences. The main one is the presence of the light η' state, which in the quenched QCD (QQCD) does not decouple from the octet of pseudoscalar mesons. As a result the "pion" propagator does not exhibit only the pole structure but also the double-pole one, which is the source of pathology of the quenched approximation.

2.1 Quenched chiral Lagrangian

In QQCD, besides the quarks q_a ($a = 1, 2, 3$), one also has the bosonic "ghost" quarks $\tilde{q}_a$, of spin $\frac{1}{2}$ and with identical mass, $m_{\tilde{q}_a}/m_{q_a} = 1$. Their rôle is to cancel the contributions of the closed quark loops, i.e. they provide quenching. If one assumes that the symmetry breaking pattern of QQCD is similar to the one of the full QCD, i.e. the graded $SU(3|3)_L \otimes SU(3|3)_R$ spontaneously breaks down to $SU(3|3)_V$, the following chiral Lagrangian can be written

$$\mathcal{L}_{\text{light}} = f^2 8\,\mathrm{str}\left(\partial_\mu \Sigma \partial^\mu \Sigma^\dagger\right) + f^2 \mu_0 2\,\mathrm{str}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) + \alpha_0 \partial_\mu \Phi_0 \partial^\mu \Phi_0 - m_0^2 \Phi_0^2 + \mathcal{L}_4 \;, \quad (1)$$

where we adopt the convention that $f \approx 130$ MeV, and the notation

$$\Sigma = \exp\left(2i\frac{\Phi}{f}\right), \qquad \Phi = \begin{pmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{pmatrix} \;. \quad (2)$$

The following comments are in order:

- besides the standard ($q\bar{q}$) Goldstone bosons (π, K, η) and the η' meson [3], organised in the 3×3 matrix

$$\phi = \begin{pmatrix} 1\sqrt{2}\pi^0 + 1\sqrt{6}\eta + 1\sqrt{3}\eta' & \pi^+ & K^+ \\ \pi^- & -1\sqrt{2}\pi^0 + 1\sqrt{6}\eta + 1\sqrt{3}\eta' & K^0 \\ K^- & \bar{K}^0 & -2\sqrt{6}\eta + 1\sqrt{3}\eta' \end{pmatrix} \;, \quad (3)$$

  the ghost–ghost ($\tilde{q}\bar{\tilde{q}}$) bosons ($\tilde{\phi}$), as well as the pseudoscalar fermions $\bar{\tilde{q}}q$ ($\chi^\dagger$) and $\bar{q}\tilde{q}$ (χ) also appear in eq. (2);

- the global symmetry $SU(3|3)_L \otimes SU(3|3)_R$ is graded and in eq. (1), instead of the familiar trace, one deals with the supertrace, $\mathrm{str}(\Phi) = \mathrm{tr}(\phi) - \mathrm{tr}(\tilde{\phi})$;

- as already stressed, η' does not decouple from the light pseudoscalar octet. Its effect is included in two terms of the Lagrangian (1), each of them multiplied by a new low energy constant, namely α_0 and m_0. Note that $\Phi_0 = \mathrm{str}(\Phi)/\sqrt{6} = (\eta' - \tilde{\eta}')/\sqrt{2}$;

[2] For an elegant alternative way to introduce the partially quenched ChPT, see ref. [12].

[3] We will neglect the mixing of η and η' states, as it is irrelevant for the discussion that follows.

– the quark–ghost mass matrix is diagonal $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s, m_u, m_d, m_s)$. After diagonalizing the mass term in eq. (1), one gets

$$m_\pi^2 = 4\mu_0 m_q \ , \quad m_K^2 = 2\mu_0 \left(m_q + m_s\right) \ , \quad m_\eta^2 = 4\mu_0 3 \left(m_q + 2m_s\right) \ , \tag{4}$$

verifying the familiar Gell-Mann–Okubo relation, $4m_K^2 - m_\pi^2 - 3m_\eta^2 = 0$. Notice that we neglect the isospin symmetry breaking, i.e. we set $m_u = m_d \equiv m_q$;

– in eq. (1) $\mathcal{L}_4$ stands for the terms of $\mathcal{O}(p^4)$, of which we write only those that are relevant to the heavy-to-light form factors, namely

2.2 Incorporating the heavy quark symmetry

QChPT has been combined with the leading order HQET in the works by Booth [15] and by Sharpe and Zhang [16]. They applied the approach to compute the heavy–light decay constants, the B^0–$\bar{B}^0$ mixing parameter and the Isgur–Wise function [4].

To devise a Lagrangian for the heavy–light mesons, it is necessary to include the heavy quark spin symmetry. This is achieved by combining the pseudoscalar (P^a) and vector ($P_\mu^{*\ a}$) heavy–light mesons in one field:

$$H_a(v) = 1 + \not{v} 2 \left[P_\mu^{*\ a}(v)\gamma_\mu - P^a(v)\gamma_5\right] \ , \tag{5}$$

where $(1+\not{v})/2$ projects out the particle component of the heavy quark only. The conjugate field is defined as $\overline{H}_a(v) = \gamma_0 H_a^\dagger(v)\gamma_0$, whereas the covariant derivative and the axial field have the following forms:

$$\begin{aligned} D_{ba}^\mu H_b &= \partial^\mu H_a - H_b \mathbf{V}_{ba}^\mu = \partial^\mu H_a - H_b 12[\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger]_{ba} \ , \\ \mathbf{A}_\mu^{ab} &= i2[\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger]_{ab} \ . \end{aligned} \tag{6}$$

In the above equation, a and b run over the light quark flavours and $\xi = \sqrt{\Sigma}$. With these ingredients in hand, we now write the quenched chiral Lagrangian for the heavy–light mesons as

$$\begin{aligned} \mathcal{L}_{\text{heavy}} = &- \mathrm{str}_a \, \mathrm{Tr}[\overline{H}_a iv{\cdot}D_{ba} H_b] + g \, \mathrm{str}_a \, \mathrm{Tr}[\overline{H}_a H_b \gamma_\mu \mathbf{A}_{ba}^\mu \gamma_5] \\ &+ g' \, \mathrm{str}_a \, \mathrm{Tr}[\overline{H}_a H_a \gamma_\mu \gamma_5] \, \mathrm{str}(\mathbf{A}^\mu) + \mathcal{L}_3 \ , \end{aligned} \tag{7}$$

where g (g') is the coupling of the heavy meson doublet to the Goldstone boson (to η' or $\tilde{\eta}'$). A term with g' is thus an artefact of the quenched theory. The higher order terms in the expansion in $v{\cdot}p \sim \mathcal{O}(p)$, $\mathcal{O}(p^2)$, and in $m_q \sim \mathcal{O}(p^2)$, denoted as $\mathcal{L}_3$ in eq. (7), have the following form [15]

$$\mathcal{L}_3 \ = \ 2\lambda_1 \, \mathrm{str}_a \, \mathrm{Tr}[\overline{H}_a H_b](\mathcal{M}_+)_{ba} + k_1 \, \mathrm{str}_a \, \mathrm{Tr}[\overline{H}_a iv{\cdot}D_{bc} H_b](\mathcal{M}_+)_{ca}$$

[4] For a recent result on the Isgur–Wise function in partially QChPT, see ref. [17].

$$+k_2\,\mathrm{str}_a\,\mathrm{Tr}[\overline{H}_a iv\cdot D_{ba} H_b]\,\mathrm{str}_c(\mathcal{M}_+)_{cc} + \dots\ , \tag{8}$$

with $\mathcal{M}_+ = (1/2)(\xi^\dagger M \xi^\dagger + \xi M \xi)$. We displayed only the terms that contribute to the heavy-to-light form factors. In the above equations, "Tr" stands for the trace over Dirac indices, whereas "str" is the supertrace over the light flavour indices.

As in the previous subsection, one can easily verify that after replacing str $\to$ tr, $\Phi \to \phi$, $\eta' \to 0$ and $g' \to 0$, one recovers the standard chiral Lagrangian for heavy–light mesons (for recent reviews see ref. [18], and for the original papers see ref. [19]).

2.3 Form factors

A frequently encountered decomposition of the matrix elements relevant to the leptonic, the semileptonic and the penguin-induced hadronic matrix elements is

$$\begin{aligned}
&\langle 0|\bar q\gamma_\mu\gamma_5 b|B(p_B)\rangle = i f_B p_{B\mu}\ ,\\
&\langle P(p)|\bar q\gamma_\mu b|B(p_B)\rangle = \big((p_B+p)_\mu - q_\mu m_B^2 - m_P^2 q^2\big)\, F_+(q^2) + m_B^2 - m_P^2 q^2 q_\mu F_0(q^2)\ ,\\
&\langle P(p)|\bar q\sigma_{\mu\nu} q^\nu b|B(p_B)\rangle \ = \ i\left(q^2(p_B+p)_\mu - (m_B^2 - m_P^2) q_\mu\right)\ F_T(q^2)\ m_B + m_P\ ,
\end{aligned} \tag{9}$$

where $q = d$ or s, $|P(p)\rangle$ is the light pseudoscalar meson state (pion or kaon) and $q^\nu = (p_B - p)^\nu$.

We will be working in the static limit, $m_B \to \infty$. The eigenstates of the QCD and HQET Lagrangians are related as

$$\lim_{m_B\to\infty} 1\sqrt{m_B}|B(p_B)\rangle_{\rm QCD} = |B(v)\rangle_{\rm HQET}\ . \tag{10}$$

In the static limit it is more convenient to use definitions in which the form factors are independent of the heavy meson mass, namely

$$\begin{aligned}
&\langle 0|\bar q\gamma_\mu\gamma_5 b_v|B(v)\rangle_{\rm HQET} = i\hat f v_\mu\\
&\langle P(p)|\bar q\gamma_\mu b_v|B(v)\rangle_{\rm HQET} = \big[p_\mu - (v\cdot p)v_\mu\big]\, f_p(v\cdot p)\ +\ v_\mu f_v(v\cdot p)\ ,
\end{aligned} \tag{11}$$

where the field b_v does not depend on the heavy quark mass [20]. The form factors $f_{p,v}$ are functions of the variable

$$v\cdot p = m_B^2 + m_P^2 - q^2 2m_B\ , \tag{12}$$

which in the heavy meson rest frame is the energy of the light meson E_P. The relation between the quantities defined in eqs. (9) and (11) is obtained by matching QCD to HQET at the scale $\mu \sim m_b$ [21]:

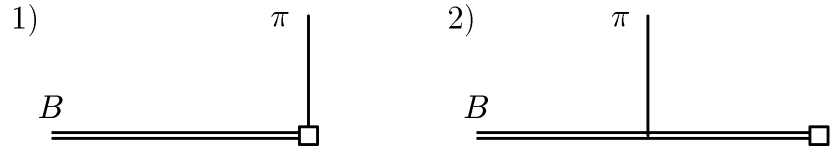

Figure 1: The point 1) and the pole 2) tree level Feynman diagrams contributing to *heavy* → *light* transition form factors. The box denotes the weak current insertion.

2.4 Heavy–light current

Having in mind eq. (**??**), we only need to consider the $(V-A)$ Dirac structure. In the static heavy quark limit and at next-to-leading order in the chiral expansion, the bosonized *heavy* → *light* current reads [15]

$$J^\mu \equiv \bar{q}_a \gamma^\mu (1-\gamma_5) Q \quad \to \tfrac{i\alpha}{2}\,\mathrm{Tr}[\gamma^\mu(1-\gamma_5)H_b]\xi^\dagger_{ba}[1+V_L'(0)\Phi_0] \tag{13}$$

$$+\tfrac{i\alpha}{2}\varkappa_1\,\mathrm{Tr}[\gamma^\mu(1-\gamma_5)H_c]\xi^\dagger_{ba}(\mathcal{M}_+)_{cb} \tag{14}$$

$$+\tfrac{i\alpha}{2}\varkappa_2\,\mathrm{Tr}[\gamma^\mu(1-\gamma_5)H_b]\xi^\dagger_{ba}\,\mathrm{str}_c(\mathcal{M}_+)_{cc}\ .$$

(15)When bosonizing the current J^μ, one could envisage an arbitrary function $V_L(\Phi_0)$ in the first line of eq. (13), expanded in a Taylor series. Only the terms linear in Φ_0 are relevant to our purpose and $V_L(0)$ can be normalised to 1 [16]. The appearance of $V_L'(0)$ is yet another artefact of the quenched theory. The phase of the heavy meson can be chosen in such a way that $V_L'(0)$ is completely imaginary, whereas the constants α, $\varkappa_1$ and $\varkappa_2$ are real. Notice that at leading order in the chiral expansion, the constant α is simply the heavy–light meson decay constant in the static limit ($m_Q \to \infty$), i.e. $\hat{f}$ in eq. (11).

3 Chiral Corrections to $f_p(v{\cdot}p)$ and $f_v(v{\cdot}p)$

In this section we explain the main steps undertaken to compute the chiral logarithmic corrections for $\hat{f}$, $f_p(v{\cdot}p)$ and $f_v(v{\cdot}p)$. Our results for the decay constant $\hat{f}$ agree with those of refs. [15, 16]. They will be used in section 5.3 to construct the ratios that are independent of counter-terms. In ref. [24] ChPT has been applied to compute the heavy-to-light form factors. We repeat that calculation and extend it to the quenched case.

3.1 $f_p^{\mathrm{TREE}}(v{\cdot}p)$ and $f_v^{\mathrm{TREE}}(v{\cdot}p)$

We start the discussion by writing the tree level expressions:

$$f_p(v{\cdot}p) = \alpha f g v{\cdot}p + \Delta_i^*\ , \qquad f_v(v{\cdot}p) = \alpha f\ , \qquad \hat{f} = \alpha\ , \tag{16}$$

The point and the pole diagrams, depicted in fig. 1, describe f_v and f_p, respectively. The index "i" in $\Delta_i^* = m_{B_i^*} - m_{B_i}$ labels the light quark flavour. When necessary, we will use P_{ij} to denote the light pseudoscalar mesons with the valence quark content $q_i\bar{q}_j$.

Although the heavy quark spin symmetry suggests $\Delta_i^* \to 0$, we will keep this term finite because it provides the pole at $m^2_{B_i^*}$ to the form factor f_p, or equivalently to the form factors $F_{+,T}$ [5]. The form factor f_v (or F_0), on the other hand, does not depend on $(v{\cdot}p)$ at the tree level.

3.2 $f_p^{1-\mathrm{loop}}(v{\cdot}p)$ and $f_v^{1-\mathrm{loop}}(v{\cdot}p)$

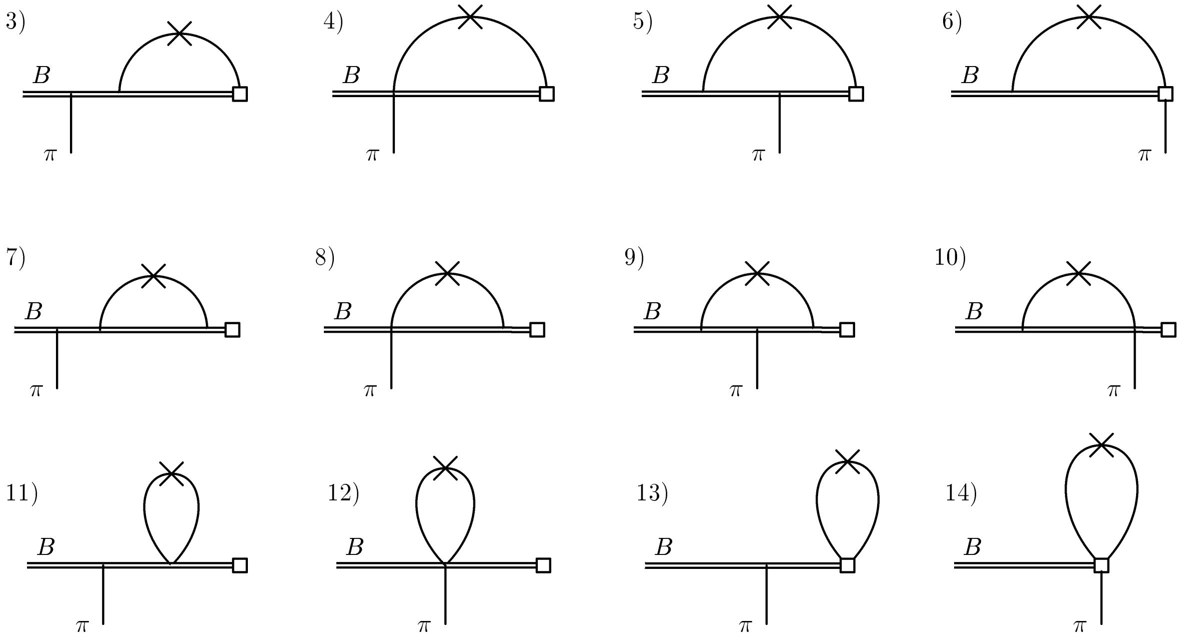

Figure 2: The one loop contributions to the $B \to \pi$ transition. Double/single lines denote the heavy/light meson, while the weak current insertion is depicted by the empty box. Crosses stand for the m_0 hairpin vertex (1). Each graph represents two Feynman diagrams in the quenched ChPT: a) the diagram without the hairpin vertex (cross) and b) the same diagram with a cross. In full ChPT only the diagrams without the cross are present. The amplitudes corresponding to the diagrams are listed in appendix B.

Neglecting the crosses, all the graphs shown in fig. 2 describe the one loop chiral contributions to the form factors $f_{v,p}(v{\cdot}p)$ in the full ChPT. In the quenched theory instead, the graphs both with and without crosses appear. The cross denotes the so-called hairpin vertex, i.e. the dipole term in the "pion" propagator:

$$1p^2 - m_P^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \alpha_0 p^2 - m_0^2 (p^2 - m_P^2)^2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \tag{17}$$

In the computation of the loop integrals, the naive dimensional regularisation has been used, with renormalization prescription $\overline{\mathrm{MS}} + 1$, where $\bar{\Delta} = 2/\epsilon - \gamma + \ln(4\pi) + 1$ is subtracted [13]. We neglect the isospin symmetry breaking ($m_u = m_d \equiv m_q$), as well as the

[5] The pole dominance is easily seen if one rewrites the denominator of $f_p(v{\cdot}p)$ as $v{\cdot}p + \Delta^* = (m_{B^*}/2)\left(1 - q^2/m^2_{B^*}\right)$, where the corrections $(m_{B^*} - m_B)/m_B$ and m_P^2/m_B^2 are neglected.

mass differences between B, B_s, B^*, B_s^* meson states, whenever they appear in the loop. The resulting expressions can be written as

Quenched ChPT

$$
\begin{aligned}
f_p^{B_j\to P_{ij}}(v\cdot p) &= \alpha f\ \ g v\cdot p + \Delta_i^*\ \ \left[1 + \delta f_p^{B_j\to P_{ij}} + k_1 2 m_j - 4L_5 4\mu_0 f^2 (m_i + m_j)\right]\ , \\
f_v^{B_j\to P_{ij}}(v\cdot p) &= \alpha f\ \ \left[1 + \delta f_v^{B_j\to P_{ij}} + (k_1 2 + \varkappa_1)\, m_j - 4L_5 4\mu_0 f^2 (m_i + m_j)\right]\ , \\
\hat{f}_j &= \alpha\ \ \left[1 + \delta \hat{f}_j + (k_1 2 + \varkappa_1)\, m_j\right]\ ,
\end{aligned}
\tag{18}
$$

Full ChPT

$$
\begin{aligned}
f_p^{B_j\to P_{ij}}(v\cdot p) &= \alpha f\ \ g v\cdot p + \Delta_i^*\ \ \Big[1 + \delta f_p^{B_j\to P_{ij}} + k_1 2 m_j - 4L_5 4\mu_0 f^2 (m_i + m_j) \\
&\qquad\qquad + \left(k_2 2 - 8L_4 4\mu_0 f^2\right)(m_u + m_d + m_s)\Big]\ , \\
f_v^{B_j\to P_{ij}}(v\cdot p) &= \alpha f\ \ \Big[1 + \delta f_v^{B_j\to P_{ij}} + (k_1 2 + \varkappa_1)\, m_j - 4L_5 4\mu_0 f^2 (m_i + m_j) \\
&\qquad\qquad + \left(k_2 2 + \varkappa_2 - 8L_4 4\mu_0 f^2\right)(m_u + m_d + m_s)\Big]\ , \\
\hat{f}_j &= \alpha\ \ \left[1 + \delta \hat{f}_j + (k_1 2 + \varkappa_1)\, m_j + (k_2 2 + \varkappa_2)\,(m_u + m_d + m_s)\right]\ .
\end{aligned}
\tag{19}
$$

We separated the chiral loop corrections (δf) from those involving the counter-terms. The loop corrections to the form factors are written as

$$
\delta f_{v,p}^{B_j\to P_{ij}} = \sum_I \delta f_{v,p}^{(I)} + 12\delta Z_{B_j}^{\rm loop} + 12\delta Z_{P_{ij}}^{\rm loop}\ , \tag{20}
$$

where the sum runs over all diagrams shown in fig. 2, and $\delta Z_{B_j(P_{ij})}$ encodes the chiral loop contributions arising from the wave function renormalization of the heavy–light and light-light meson respectively. The explicit expressions are lengthy and are collected in appendices B.1 (for the quenched) and B.2 (for the full theory). As can be seen from eqs. (18) and (19), no dependence on $(v\cdot p)$ arises from the counter-terms. The modification of the tree-level $(v\cdot p)$ dependence of the form factors $f_{v,p}$ is entirely due to the chiral loop corrections. It is worth noticing that in the quenched approximation the form factor $f_v^{B\to\pi}$ is completely independent of $(v \cdot p)$. This is so because the contribution from the diagram 4 in fig. 2 is vanishing in the isospin limit $m_u = m_d$.

An important feature emerging from this calculation is that the quenched form factors $f_{v,p}$ and quenched decay constant $\hat{f}$ suffer from the common quenched pathology, that is from the presence of the quenched chiral logs $\propto m_0^2 \log(m_P^2/\mu^2)$. Such terms are divergent in the limit $m_P \to 0$, suggesting that in the quenched approximation the chiral limit for any of $F_{+,0,T}^{B\to\pi}$ or f_B is not defined. This feature is also present in the light meson sector, e.g. for the chiral condensate, the light meson decay constant consisting of non-degenerate quark flavours, etc. [25, 26].

4 Choice of Parameters

For the numerical analysis of the expressions given in eqs. (18) and (19) we need to make a specific choice of quite a number of low energy constants. To do so we will mainly rely on the existing lattice data. The standard procedure in extracting the low energy constants at the 1-loop level, involves "undressing" the chiral loop effects from the desired constants (see e.g ref. [13]). Such a strategy, however, is not applicable when using the lattice data. The reason is that the direct lattice computations are made for the light pseudoscalar mesons $m_P \gtrsim 400$ MeV, and the physical results ($m_P \to m_\pi^{phys}$) are obtained through a linear or quadratic extrapolation in m_P^2. Since those extrapolations are made without including the effects of the chiral logs, their subtraction would lead to unrealistic estimates of the low energy constants. For that reason, in what follows, we adopt the approximating procedure and fix the parameters at the tree level in ChPT. In table 1 we collect the parameters whose (range of) values we were able to fix.

Coupling	Full Theory	Quenched Theory
α [GeV$^{3/2}$]	0.56 ± 0.04	0.48 ± 0.03
g	0.50 ± 0.10	0.56 ± 0.12
f [GeV]	0.130	0.124 ± 0.004
μ_0 [GeV]	1.14 ± 0.20	1.13 ± 0.04
L_4 [$\times 10^{-3}$]	$-(0.5 \pm 0.5)$	(0.0 ± 0.5)
L_5 [$\times 10^{-3}$]	0.8 ± 0.5	1.1 ± 0.3
m_0 [GeV]	–	0.64 ± 0.06
g' [GeV]	–	-0.6 to 0.6

Table 1: Low energy constants whose determination is discussed in the text.

In the following we briefly discuss each of those values.

- $\underline{\alpha}$: This constant is equal to the heavy–light decay constant in the static limit of QCD. Its value can be obtained from $f_D\sqrt{m_D}$ and/or $f_B\sqrt{m_B}$, which are then extrapolated (in inverse heavy–light meson mass) to the infinite mass limit as

$$f_H\sqrt{m_H} = \alpha\left(1 + A/m_H + B/m_H^2 + \dots\right) . \tag{21}$$

From lattice QCD and QCD sum rules, it is known that the slope A is large and negative, whereas the value of the curvature B is small. Therefore, to a good accuracy, one can set $B = 0$ and neglect higher terms in $1/m_H$. In this way we obtain:

- ∗ From extensive quenched lattice simulations by the MILC collaboration, one can deduce $\alpha^{\rm quench} = 0.45(5)$ GeV$^{3/2}$ [27]. In the same paper, they also present the results of their unquenched simulations, from which we extract $\alpha^{\rm full} = 0.53(7)$ GeV$^{3/2}$. These numbers agree quite well with the values obtained by CP-PACS, namely $\alpha^{\rm quench} = 0.50(4)$ GeV$^{3/2}$, and $\alpha^{\rm full} = 0.57(6)$ GeV$^{3/2}$ [28]. Notice that both references use the same treatment of the heavy quark on the lattice (so called Fermilab formalism).
- ∗ UKQCD [29] and APE [30] compute heavy–light meson decay constants using the fully relativistic lattice QCD, but only for the mesons of masses $m_H \in (1.8, 2.7)$ GeV. From the linear fit of the form (21), from their quenched data UKQCD obtain [6] $\alpha^{\rm quench} = 0.49(4)$ GeV$^{3/2}$, in agreement with the previous result by APE, $\alpha^{\rm quench} = 0.48(5)$ GeV$^{3/2}$.
- ∗ Recent results obtained by using the QCD sum rules agree quite well with the above unquenched values. From the compilation of the QCD sum rule estimates in ref. [31], one has $\alpha^{\rm full} = 0.58(9)$ GeV$^{3/2}$ (for the most recent result see [32]).

◦ $\underline{g}$: This constant is related to the phenomenological coupling $g_{D^*D\pi}$ as

$$g_{D^*D\pi} = 2\sqrt{m_D m_{D^*}} f_\pi g \,, \tag{22}$$

where one can also set $m_{D^*} = m_D$ because the above relation is defined only in the static limit, in which the heavy quark spin symmetry is exact. From the experimentally measured total width of the D^{*+}-meson, one gets $g_c^{\rm full} = 0.59(7)$ [33]. The subscript "c" warns us that the value is obtained in the charm quark mass sector. On the lattice, that value has been computed recently in the quenched approximation, leading to $g_c^{\rm quench} = 0.66(9)$ [34]. To get the value of g one needs to extrapolate to the infinite heavy quark mass limit, i.e. $g_Q = g + \gamma/m_H + \dots$ Lattice data of ref. [34] suggest that the slope γ is negligible, and thus $g/g_c \approx 1$. On the other hand, the LCSR calculation suggests that the same slope is significant and positive [35]. By neglecting the terms of $\mathcal{O}(1/m_H^2)$ and higher, ref. [35] leads to $g/g_c \approx 0.7$. We will take the average of the two (lattice and LCSR), that is $g/g_c \approx 0.85$, and add the difference in the error bar of both quenched and unquenched values [7].

◦ $\underline{f}$: To get the chiral coupling constant we will use $f_\pi = 132$ MeV and $f_K = 160$ MeV, and the fact that $m_s/m_q = 24.4 \pm 1.5$ [37]. After linearly extrapolating to the chiral limit ($m_q \to 0$), we get $f = 130$ MeV. The recent extensive quenched lattice study with Wilson fermions gives $f = 119{\pm}7$ MeV [38], whereas

[6] We thank G. Lacagnina for communicating this result to us.

[7] We are aware of the result of ref. [36], $g = 0.42(9)$, where this coupling has been computed for the first time on the lattice, in the static HQET. However, in view of the insignificant statistics, very coarse lattice and only two light quark masses, the value $g = 0.42(9)$ should be considered as exploratory. The improved calculation of g, along the lines described in ref. [36] would be most welcome, though.

the one with staggered fermions gave $f = 125 \pm 9$ MeV [39]. The latter result has also been obtained on the larger lattices with domain wall fermions, namely $f = 125 \pm 7$ MeV [40]. As a weighted average, we will take $f = 124 \pm 4$ MeV.

- $\underline{\mu_0}$: From the Gell-Mann–Oakes–Renner (GMOR) formula [41], it is easy to identify $\mu_0 = -\langle \bar{q}q \rangle / f_\pi^2$. Its value (in the full theory) can be fixed by using $\langle \bar{q}q \rangle^{\overline{\rm MS}}(2\ \text{GeV}) = -(267(16)\ \text{MeV})^3$ [42]. The quenched value is extracted from the lattice data. Recent results in the $\overline{\rm MS}$ scheme (at the renormalization scale 2 GeV) are $\mu_0 = 1.13(4)$ GeV [38], and $\mu_0 = 1.10(11)$ GeV [43]. With these numbers and by using $f = 124 \pm 4$ MeV, the respective numerical values for the chiral condensate in the quenched approximation are $\langle \bar{q}q \rangle^{\overline{\rm MS}}(2\ \text{GeV}) = -(259(6)\ \text{MeV})^3$, and $\langle \bar{q}q \rangle^{\overline{\rm MS}}(2\ \text{GeV}) = -(257\ (10)\ \text{MeV})^3$.

- $\underline{L_{4,5}}$: These two couplings have been extensively discussed in the literature [14]. Their values in the full theory, at the scale $\mu \simeq 1$ GeV, are given in table 1. In quenched QCD, the coupling L_5 has been recently computed in ref. [44], with the result $\alpha_5 = 0.99(26)$, obtained at the scale $\mu = 1.17$ GeV. That value corresponds to $L_5 = \alpha_5/(128\pi^2)$, which we then run to $\mu = 1$ GeV, to get the number given in table 1. On the other hand, the quenched estimate of L_4 is not available. On the basis of the large N_c expansion we only know that L_4 is smaller than L_5 (see ref. [13]). We will take it to be zero and vary by $\pm 0.5 \times 10^{-3}$.

- $\underline{m_0}$: This mass characterizes the magnitude of the hairpin insertion (crosses in fig. 2). It enters in the coefficient of the quenched chiral log, which, in the literature, is often referred to as $\delta = m_0^2/(24\pi^2 f_\pi^2)$. The precise value of δ is unknown at present, mainly because in realistic lattice studies it is very difficult (if at all possible!) to resolve the finite lattice volume effects from the quenched chiral logs. This is why it is still not completely clear whether the observed deviations from a linear dependence of the pseudoscalar meson squared (m_P^2) on the light quark masses (m_q) is due to the presence of the chiral logs, or if it is an artefact of the lattice geometry (see e.g. ref. [45]). With this remark in mind, we now quote recent values for the parameter δ, as obtained from the lattice studies of the dependence of m_P^2 on m_q:

  - with Wilson fermions, $\delta = 0.10(2)$ [38];
  - with modified Wilson fermions, $\delta = 0.073(20)$ [46] [8];
  - with domain wall fermions, $\delta = 0.029(7)$ [47];
  - with overlap fermions $\delta = (0.0\text{–}0.2)$ [48] and $\delta = (0.2\text{–}0.3)$ [49].

  For the pre-1996 results see the review in [26]. The value of δ obtained by the CP-PACS collaboration stands out because they made an extensive high statistics study on large lattice volumes and extrapolated to the continuum [38]. A worrisome aspect of this value however is that the light pseudoscalar mesons that they access directly lie in the range $m_P \in (300, 750)$ MeV, for which the

[8] The final result of ref. [46], $\delta = 0.065(13)$, is obtained by combining various ways of extracting this quantity from the lattice data. For an easier comparison with other groups, we quote the result given in eq. (29) of ref. [46], which is obviously fully consistent with their final value.

significance of the chiral logs may be questionable. Keeping this comment in mind and using their $\delta = 0.10(2)$, we get $m_0 = 0.64(6)$ GeV.

- $\underline{g'}$: The value of this constant can be obtained from $g_{BB^*\eta'}$ and $g_{DD^*\eta'}$, couplings of the lowest lying heavy–light meson doublet to the light quenched η' state. Such a lattice study has not been made so far. To get a rough estimate we may rely on the large N_c limit from which one expects $|g'| < g$.

As for the other parameters appearing in the quenched expressions, in the numerical analysis, we will first set them to zero and then vary their values in the ranges suggested by the large N_c expansion. More specifically, we take $|V_L'| \leq 0.5$, $|\varkappa_1|$, $|k_1| \leq 32\mu_0 L_5/f^2$, and $|\varkappa_2| = |k_2| = 0$. The parameter α_0 will be varied in $|\alpha_0| < 0.1$. It is important to stress that the effect of the variation of these latter parameters is completely negligible for our numerical estimates.

5 Quenching Errors

In this section we will use the expressions for the form factors (18) and (19), the parameters listed in table 1, to get an estimate of the quenching errors. We reiterate that the results of this section refer to the zeroth order in $1/m_H$ expansion and to the leading logarithmic chiral corrections, with the specific choice of parameters discussed in the previous section. We also stress that in all the following discussion we will keep the strange quark mass fixed to its quenched value of $m_s^{\overline{\rm MS}}(2~{\rm GeV}) = 105$ MeV, and $m_s^{full}/m_s^{quench} \simeq 0.85$ [50]. To examine the quenching errors we will study the following ratios

$$Q_{p,v}(v{\cdot}p, m_q/m_s) = f_{p,v}^{full}(v{\cdot}p) - f_{p,v}^{quench}(v{\cdot}p) f_{p,v}^{full}(v{\cdot}p)\,. \tag{23}$$

The parameter that makes the strongest impact on the results for $Q_{p,v}$ is g'. As stated above, its value is expected to lie between $-g \leq g' \leq g$. Since even its sign is not known we will distinguish between the two "extreme" cases, $g' = -g$ and $g' = +g$.

5.1 $B \to \pi$ transition

To examine the quenching errors for $B \to \pi$ decay we cannot set the pion mass to its physical value, because the spurious quenched chiral logs would become dominant. However, we cannot go far away from the chiral limit either, because the (Q)ChPT approach then becomes inadequate. To those two competing requirements, we should add a third: a desire to be sufficiently close to the region of quark masses (i.e. pseudoscalar meson masses squared) probed by the current quenched lattice studies [5–9]. It is not clear whether the mass interval for which all of the above requirements are satisfied exists or not. We will assume that they are satisfied for $m_P \approx 330$ MeV. From eq. (4), a "pion" of mass $m_P \approx 330$ MeV is composed of two quarks of mass corresponding to $r = m_q/m_s \simeq 0.25$, w.r.t. the strange quark mass m_s. In fig. 3 we plot the ratios $Q_{p,v}(v{\cdot}p, r = 0.25)$.

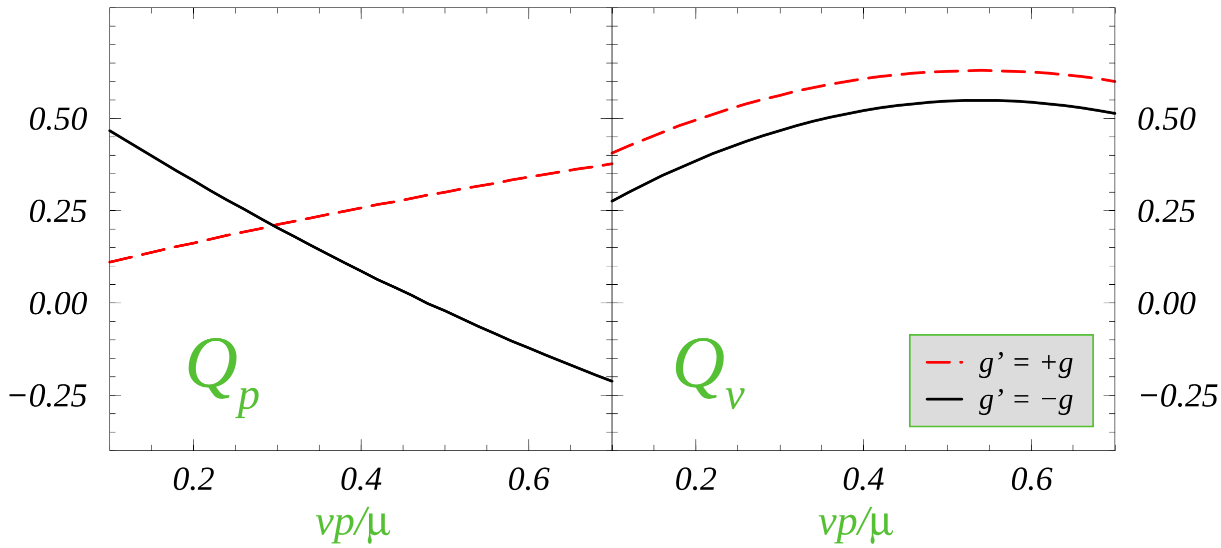

Figure 3: The ratios $Q_{p,v}(v\cdot p)$ are defined in eq. (23). The plots refer to the mesons consisting of two degenerate (valence) quarks with mass $m_q = r\ m_s$ and $r = 0.25$. The counter-term coefficients $k_{1,2}$, $\varkappa_{1,2}$, as well as $|V_L'(0)|$ are neglected, while $\mu = 1$ GeV $\simeq \Lambda_\chi$. Values of other parameters are given in table 1.

We notice that regardless of the value chosen for g', the quenching errors on the form factor $f_p(v\cdot p)$ are not excessively large. This is important since this form factor is the only one entering the $B \to \pi\ell\nu$ decay rate, from which we hope to be able to extract the value for $|V_{ub}|$. By specifying $g' = +g$, we observe that $Q_p(v\cdot p) > 0$, for any $(v\cdot p)/\mu < 0.7$. In other words, quenched values for the form factors are smaller than the unquenched ones. This remains true for all $r \geq 0.2$.

If instead we take $g' = -g$, the ratio $Q_p(v\cdot p)$ has a zero. Putting it differently, $\exists (v\cdot p)/\mu < 0.7$, so that the quenching errors vanish, $f_p^{full} = f_p^{quench}$. For $r = 0.25$ the point of zero quenching errors is found for $(v\cdot p/\mu) = 0.47(3)$. After identifying such points for $r = m_q/m_s \in [0.1, 1]$ (or equivalently for $m_P \in (220, 700)$ MeV), we fit them to a quadratic form and obtain

$$v\cdot p\mu = 0.25(3) + 1.0(2)r - 0.8(1)r^2 \qquad (g' = -g)\ . \tag{24}$$

We repeat that this result is valid for the parameters whose values are varied within the ranges discussed in the previous section. In fig. 4, we show the curve in the plane $Q_p(v\cdot p, r) = 0$. In the same plot we also show the curves corresponding to $Q_p(v\cdot p, r) = \pm 10$ %. While for $g' = -g$ such a band really exists, for $g' = +g$ only a part corresponding to $Q_p(v\cdot p, r) = +10\%$ can be reached, for $r < 0.4$. We have

$$v\cdot p\mu = 0.4(1) - 2.1(1)r + 2.5(3)r^2\ , \qquad (g' = +g). \tag{25}$$

In summary, it is possible to find combinations of the pion mass and the pion recoil energy such that the quenching errors on the dominant form factor $f_p(v\cdot p)$ are kept under the 10% level.

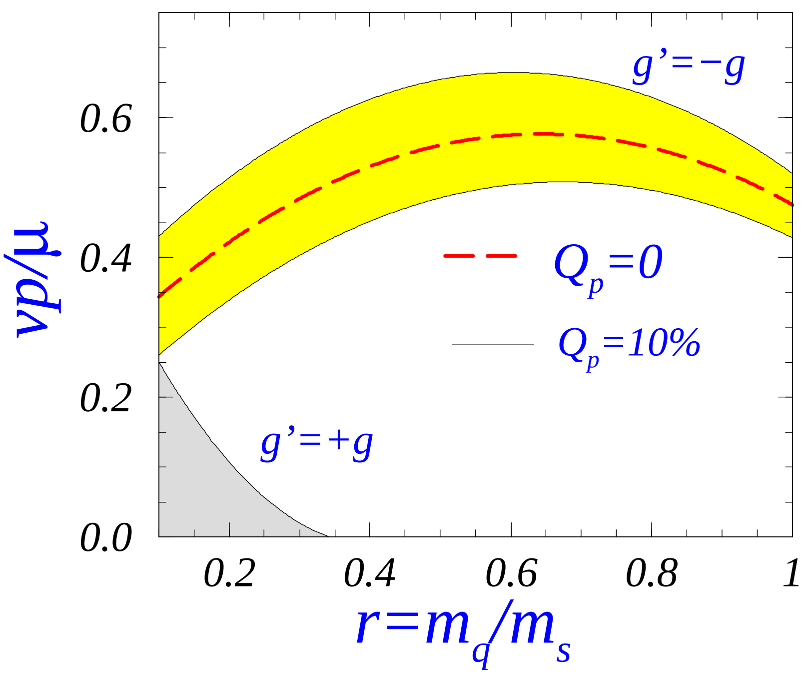

Figure 4: The thick dashed line corresponds to the zero quenching errors ($Q_p(v{\cdot}p, r) = 0$) in the case when $g' = -g$ (such a line is not accessible for $g' = +g$). The shaded region correspond to the variation of $Q_p(v{\cdot}p, r)$ by 10%. Notice that in the case $g' = -g$, the upper (lower) curve corresponds to $Q_p(v{\cdot}p, r)$ equal to -10% ($+10\%$). For the case $g' = +g$, only the region $Q_p(v{\cdot}p, r) < +10\%$ is shown.

From fig. 3 we also see that for our referential $r = 0.25$, there exists a point $(v{\cdot}p) \approx 0.3$ such that the ratio Q_p is independent of the value of g'. At that point, we get

$$Q_p(0.3\mu, r \simeq 0.25) \simeq 20\% \,. \tag{26}$$

This result is a useful estimate of the quenching error for a realistic values of $(v{\cdot}p)$ and r, which are currently probed on the lattice. For the reader's convenience, we have fitted the points corresponding to $(\partial f_p/\partial g') = 0$ to a polynomial in $r \in [0.1, 1]$ and obtained

$$v{\cdot}p\mu = 0.17(1) + 0.8(1)r - 1.3(2)r^2 + 0.58(2)r^3 \,. \tag{27}$$

We also checked that for $0.1 < r \lesssim 0.35$, with r and $(v{\cdot}p)$ satisfying (27), the quenching errors are kept under the 25% level. This concludes our discussion of the form factor f_p.

The situation with the form factor $f_v(v{\cdot}p)$ is much worse. From fig. 3, we see that the quenching errors are in the range 30 to 60%, and drop below that level only for larger recoils for which the present approach is not appropriate. The quenching errors remain large when varying r in the range $r \in [0.1, 1]$. A somewhat less pessimistic situation is present at zero recoil $(v{\cdot}p = 0)$ where, contrary to the case of f_p, the form factor f_v *can* be extracted from the lattice data. For $m_P = 330$ MeV, at zero recoil we get

$$Q_v(0, r = 0.25) \simeq \begin{cases} 8\% & (g' = +g) \\ \\ 23\% & (g' = -g) \end{cases} .$$

Once again, we warn the reader that the numerical estimates made in this section are obtained for the set of low energy constants specified in table 1.

5.2 $B \to K$ transition

We now turn to the case of a kaon in the final state. In the discussion we shall fix one of the valence quarks to the strange quark mass, and vary the other one ($m_q = r\ m_s$) in the range $1/25 \lesssim r < 0.5$. In this way the "kaon" mass is varied between $m_K^{phys} < m_P < 600$ MeV. For simplicity we consider $(v{\cdot}p)/\mu \simeq 0.3$ (reasonably small recoil), and examine the quenching ratios

$$Q^K_{p,v}(0.3\mu, r) = f^{full}_{p,v}(v{\cdot}p) - f^{quench}_{p,v}(v{\cdot}p) f^{full}_{p,v}(v{\cdot}p) \,. \tag{28}$$

In the quenched expressions for $f_{p,v}$, given in appendix B, we set for the active quark $M_i^2 = 4\mu_0 m_s$, while for the spectator one we take $M_j^2 = 4\mu_0 m_s r$ (see eqs. (46), (47)). As in the previous subsection, in fig. 5 we plot $Q^K_{p,v}$ for the two "extreme" scenarios, namely $g' = +g$ and $g' = -g$. We again observe that, regardless of the value for g', the quenching errors on the form factor $f_v(v{\cdot}p)$, $Q^K_v > 0.5$. The quenching errors Q^K_p, are only moderately large if $g' = -g$, and are large for $g' = +g$.

Therefore, from this and the preceding subsection, we see that in both channels, $B \to \pi$ and $B \to K$, our approach suggests that the quenching errors on the form factor $f_v(v{\cdot}p)$

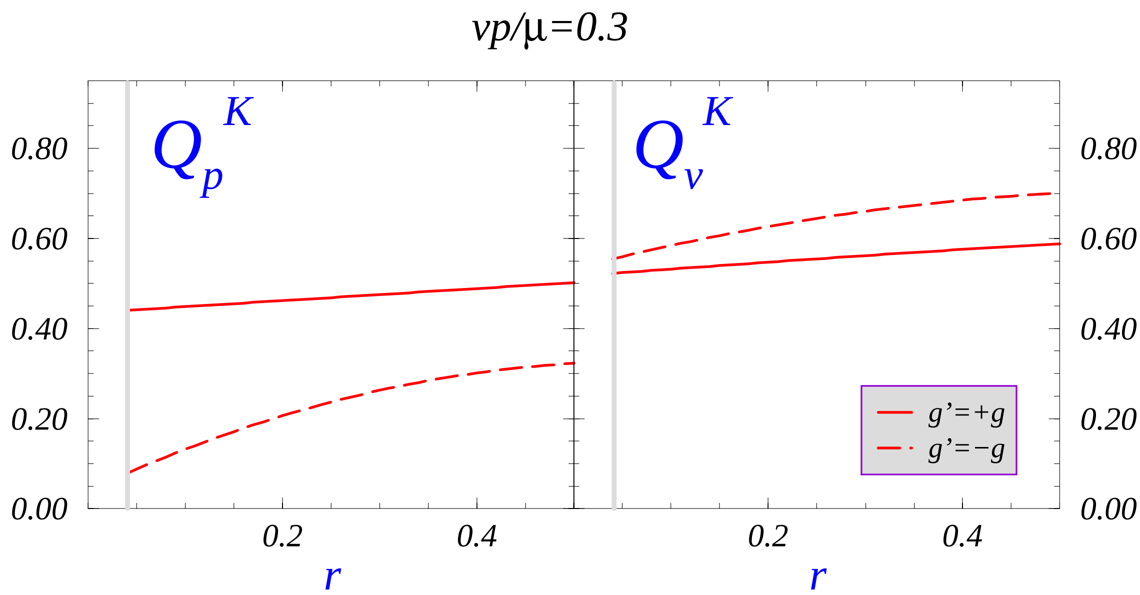

Figure 5: The ratios $Q_{p,v}(v{\cdot}p)$ from eq. (28). The plots refer to the mesons whose one valence quark is fixed to the s-quark mass, while the other has a value $m_q = rm_s$. The thick grey vertical line marks $r \simeq 1/25$ for which the physical kaon mass is reached. As before, the counter-term coefficients $k_{1,2}$, $\varkappa_{1,2}$, as well as $V_L'(0)$ are neglected, while $\mu = 1$ GeV.

(i.e. $F_0(q^2)$) are large $\gtrsim 30\%$. The quenching errors on $f_p(v{\cdot}p)$ (i.e. $F_{+,T}(q^2)$), on the other hand, depend crucially on the value of the low energy constant g': they are large for $g' > 0$, and *"not so large"* for $g' < 0$.

Before closing this subsection let us mention that in the quenched lattice studies the kaon is usually considered as a composite state of two degenerate quarks. Using $m_P^2 = 4\mu_0 m_s r$, one varies r in order to reach the physical kaon mass $m_P = m_K^{phys}$, which occurs for $r \simeq 0.5$. One may wonder if the form factors with such a kaon differ from the ones in which the quarks in the kaon are non-degenerate, with one of the quarks fixed to the strange mass and the other varying towards $r \simeq 1/25$ (i.e. the physical kaon mass). To keep the mass of the pseudoscalar meson the same in both situations, we will vary $r_{deg.} \in [0.6, 0.8]$ and $r_{ndg.} = 2r_{deg.} - 1 \in [0.2, 0.6]$. As before, we take $(v{\cdot}p/\mu) = 0.3$ and examine the following ratio

$$R_{v,p}(r_{deg.}) = f_{v,p}^{ndg.}(v{\cdot}p, r_{ndg.}) - f_{v,p}^{deg.}(v{\cdot}p, r_{deg.}) f_{v,p}^{ndg.}(v{\cdot}p, r_{ndg.})\Big|_{r_{ndg.}=2r_{deg.}-1} \tag{29}$$

in the quenched theory. From fig. 6, we again observe the same dichotomy: for $g' = -g$ the situation is quite favourable, i.e. the errors due to degeneracy in the quark masses are very small, whereas for $g' = +g$, the form factor $f_p^{deg.}$ is highly overestimated w.r.t. $f_p^{ndg.}$.

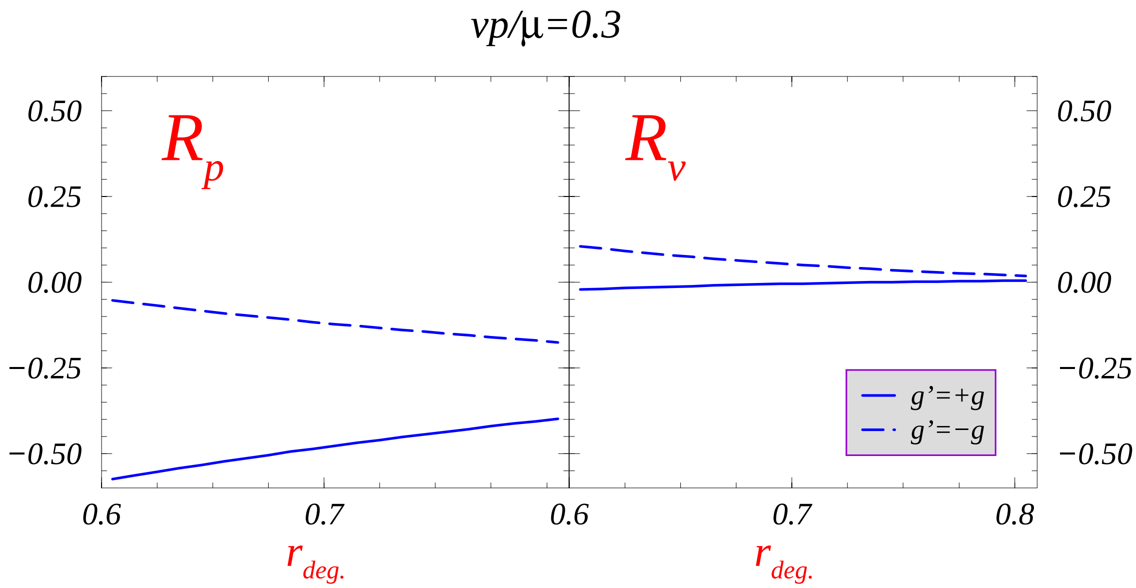

Figure 6: The ratios $R_{p,v}$ defined in eq. (29), measuring the errors induced by the degeneracy in the "kaon" quark masses in the quenched calculations .

5.3 Useful ratios

The double ratio $f_v^{B_s\to K} f_B / f_v^{B\to K} f_{B_s}$ is very gratifying from the ChPT point of view. In the double ratio of the form

$$\frac{f_{v(Q)}^{B_j\to P_{ij}}}{f_{v(Q)}^{B_i\to P_{ji}}}\,\frac{f_{B_i(Q)}}{f_{B_j(Q)}} = 1 + \delta f_{v(Q)}^{B_j\to P_{ij}} - \delta f_{v(Q)}^{B_i\to P_{ji}} + \delta f_{B_i(Q)} - \delta f_{B_j(Q)}\ , \tag{30}$$

where $B_j \sim b\bar{q}_j$ and $P_{ij} \sim q_i\bar{q}_j$, the dependence on α cancels completely and so does the counter-terms. Quenching errors on this quantity are thus far more reliably predictable in the framework of ChPT than for the separate form factors. This quantity might then prove useful in future lattice simulations with both $B_s \to P$ and $B \to P$ transitions.

A similar (partial) cancellation of dependence on counter-terms occurs also for the quenching error on the ratio $f_{p,v}^{B\to K}/f_{p,v}^{B\to\pi}$

$$\begin{aligned}\frac{f_{p,v(Q)}^{B_j\to P_{ij}}}{f_{p,v(Q)}^{B_k\to P_{lk}}} - \frac{f_{p,v}^{B_j\to P_{ij}}}{f_{p,v}^{B_k\to P_{lk}}} &= \\ &\delta f_{p,v(Q)}^{B_j\to P_{ij}} - \delta f_{p,v(Q)}^{B_k\to P_{lk}} - \delta f_{p,v}^{B_j\to P_{ij}} + \delta f_{p,v}^{B_k\to P_{lk}} - \frac{8}{f^2}\left(L_5^Q - L_5\right) m_K^2\ ,\end{aligned} \tag{31}$$

which does not depend on counter-terms if $L_5^{full} = L_5^{quench}$. In (31) we have neglected counter-terms suppressed by m_π^2/m_K^2 and assumed that δf is small (δf is independent of counter-terms by construction).

6 Chiral extrapolation in $B \to \pi$ decay

6.1 Quenched case

So far the quenched lattice studies of the $B \to \pi$ transition matrix element were restrained to the region of not very light pseudoscalar meson masses which (from the point of view of QChPT) is almost fortunate since one stays away from the region in which the spurious quenched chiral logs dominate over the other (physical) contributions. Supposing that one manages to push the lattice studies towards ever lighter "pions", the quenched chiral log will become a more important effect and one may attempt to subtract it. From the expressions presented in section 3 and in appendix B, for the very light meson $m_P < (v{\cdot}p)$, we obtain

$$
\begin{aligned}
f_p(v{\cdot}p) &= \alpha g f(v{\cdot}p + \Delta^*)\left[1 + (c_\ell^Q + c_\ell^p m_P^2)\ln\left(\frac{m_P^2}{\mu^2}\right) + c_0^p + c_1^p m_P + c_2^p m_P^2 + \dots\right] ,\\
f_v(v{\cdot}p) &= \alpha f\left[1 + (c_\ell^Q + c_\ell^v m_P^2)\ln\left(\frac{m_P^2}{\mu^2}\right) + c_0^v + c_2^v m_P^2 + \dots\right] .
\end{aligned} \tag{32}
$$

The quenched chiral log coefficient c_ℓ^Q is predicted by the QChPT and reads

$$
c_\ell^Q = -m_0^2(1+3g^2)6(4\pi f)^2 , \tag{33}
$$

which has the important consequence that by approaching ever lighter m_P, the form factors f_p^{quench} and f_v^{quench} *increase*. This is in contrast to the full (dynamical) theory in which the effect is the opposite, i.e. the chiral logs lower the form factors for the small masses, as we shall see in the next subsection (see fig. 7). The form factor $f_p(v{\cdot}p)$ picks also a term linear in m_P in this limit, which is yet another artefact of the quenched approximation. The accompanying coefficient reads

$$
c_1^p = -\frac{g^2 m_0^2}{(4\pi f)^2}\frac{4\pi}{3v{\cdot}p} . \tag{34}
$$

Thus, aiming at the physical result from the quenched simulation with very light pseudoscalar mesons, the main quenched artefacts need to be subtracted. Their coefficients are predicted by QChPT and are given in eqs. (33) and (34).

6.2 Unquenched case - An Illustration

The unquenched equivalents to the expressions (32) are not straightforward to derive. The main obstacle comes from the fact that instead of one mass in the integrals $I_2(M, v{\cdot}p)$ and $J_1(M, v{\cdot}p)$, one now has three masses: $M \in \{m_\pi,\ m_K$ and $m_\eta\}$. The behaviour of those integrals depends on the sign of $1-(v{\cdot}p)/M$ (see appendix A). The variation of the pion mass entails the change of m_K and m_η, which straddle around $(v{\cdot}p)$. This then changes the behaviour of the integrals I_2 and J_1. For the $B \to \pi$ transition, we have

$$
\begin{aligned}
f_p(v{\cdot}p) \;&=\; \alpha g f\,(v{\cdot}p + \Delta^*)\,\Big\{1 + 1(4\pi f)^2\,\Big[\; g^2\Big(4J_1(m_\pi, v{\cdot}p) + 3J_1(m_K, v{\cdot}p) + 23J_1(m_\eta, v{\cdot}p)\Big) \\
&\qquad -1 + 3g^2 12\Big(9m_\pi^2\log(m_\pi^2) + 6m_K^2\log(m_K^2) + m_\eta^2\log(m_\eta^2)\Big)\Big] + d_0^p + \ldots\Big\}\,, \qquad (35) \\
f_v(v{\cdot}p) \;&=\; \alpha f\,\Big\{1 + 1(4\pi f)^2\,\Big[15 - 27g^2 12m_\pi^2\log(m_\pi^2) + 1 - 3g^2 2m_K^2\log(m_K^2) \\
&\qquad - 1 + 3g^2 12m_\eta^2\log(m_\eta^2) + 2I_2(m_\pi, v{\cdot}p) + I_2(m_K, v{\cdot}p)\Big] + d_0^v + \ldots\Big\}\,,
\end{aligned}
$$

where $d_0^{v,p}$ is a constant and the ellipses stand for the higher order terms in the $m^2_{\pi,K,\eta}$ expansion.

$(v{\cdot}p)$	$a_v^{(0)}[\mathrm{GeV}^{1/2}]$	$a_v^{(1)}[\mathrm{GeV}^{-3/2}]$	$a_p^{(0)}[\mathrm{GeV}^{-1/2}]$	$a_p^{(1)}[\mathrm{GeV}^{-5/2}]$	Ref.
0.55 GeV	$2.5(2)^{+0.0}_{-0.6}$	$1.1(2)^{+0.0}_{-0.2}$	$0.9(1)^{+0.0}_{-0.2}$	$0.7(1)^{+0.0}_{-0.1}$	[6]
0.19 GeV	0.8(3)	0.3(3)	4.8(4)	3.6(4.5)	[7]

Table 2: The parameters describing the linear chiral extrapolation for the $B \to \pi$ form factors $f^{latt}_{v,p}(v{\cdot}p)$ at fixed $(v{\cdot}p)$, as indicated in eq. (36). The values are those obtained in the quenched lattice studies in refs. [6, 7].

To exemplify the impact of the chiral logs, we will now use the existing (quenched) lattice results for the $B \to \pi$ form factors presented in refs. [6] and [7], in which the chiral extrapolation has been made linearly, i.e.

$$
f^{latt}_{v,p}(v{\cdot}p) = a^{(0)}_{v,p}(v{\cdot}p) + a^{(1)}_{v,p}(v{\cdot}p)m_P^2\,, \qquad (36)
$$

The parameters $a^{(0,1)}_{v,p}$ of ref. [7] are obtained by fitting in the region of the light pseudoscalar mesons that correspond to $m_P \in (450, 800)$ MeV. The ones of ref. [6] are obtained from the fit in $m_P \in (540, 840)$ MeV. The numerical values are given in table 2 [9].

We will now assume that: (a) those results are the same in the full (unquenched) theory [10]; (b) the form (36) holds true down to a point $m_M \approx 250$ MeV or $m_M \approx 330$ MeV, where we smoothly match the lattice and ChPT results. The full ChPT form factor, given in appendix B, is used from the matching point m_M down to the physical pion mass. In other

[9]We are particularly indebted to Tetsuya Onogi for communicating these results to us.

[10]Since the purpose of the discussion in this section is to illustrate the impact of the chiral logs on the result of an extrapolation to the physical pion mass, this assumption should not worry the reader.

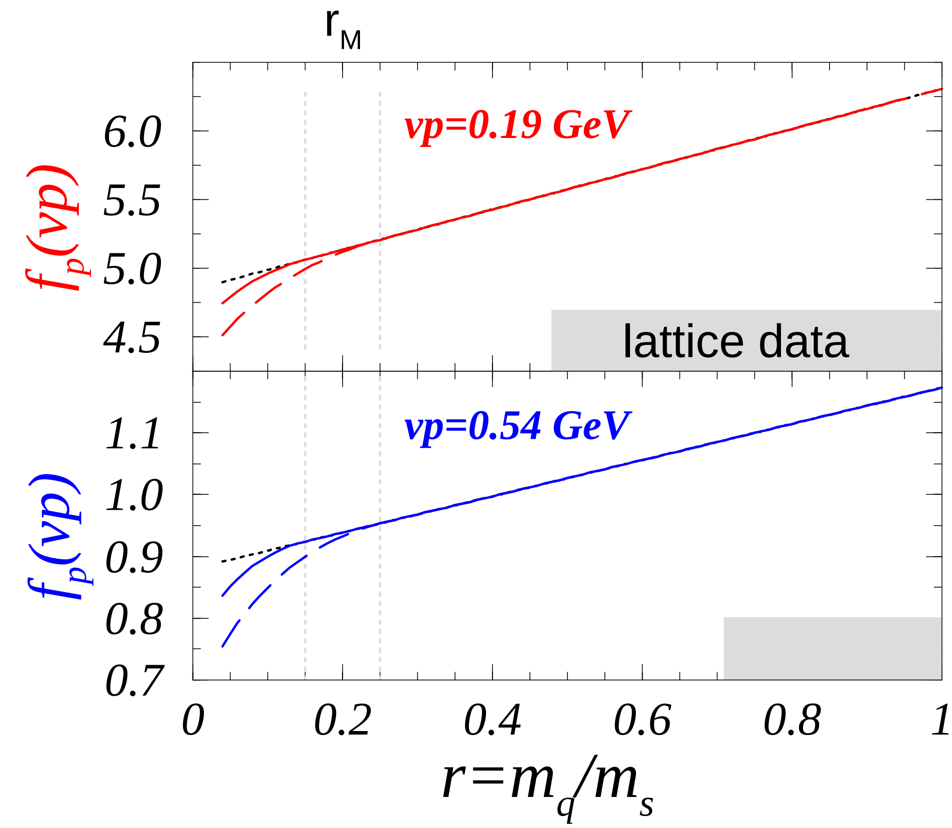

Figure 7: Illustration of the effect of inclusion of the chiral log in extrapolating to the physical pion from the masses accessed from the lattice simulations (marked by the shaded boxes). The full (dashed) curves are obtained by including the chiral logs in extrapolation starting from $r_M = 0.15$ and from 0.25 (grey vertical lines). For comparison we also show the (dotted) line corresponding to the linear extrapolation.

words, for a fixed value of $(v{\cdot}p)$, we take

$$\begin{aligned} f_{p,v}(m_P^2) \;&=\; \theta(m_P^2 - m_M^2) f_{p,v}^{latt}(m_P^2) + \\ &\theta(m_M^2 - m_P^2)\Big[f_{p,v}^{\rm ChPT}(m_P^2) - \big(f_{p,v}^{\rm ChPT}(m_M^2) - f_{p,v}^{latt}(m_M^2)\big) \\ &\qquad - \Big(\partial f_{p,v}^{\rm ChPT}/\partial m_P^2\big|_{m_P^2=m_M^2} - a_{v,p}^{(1)}\Big)\left(m_P^2 - m_M^2\right)\Big] \,. \end{aligned} \tag{37}$$

In fig. 7 we show the effect for the form factor $f_p(v{\cdot}p)$. We observe that the form factor obtained by including the chiral logs in the extrapolation to $m_P^2 = m_\pi^2$ is smaller than the one obtained by extrapolating linearly (dotted lines in the plot). The amount of that suppression obviously depends on the value of $m_M^2 = 4\mu_0 m_s r_M$: the effect of the chiral log is smaller for smaller r_M. In our example we took $r_M = 0.15$ ($m_M \simeq 250$ MeV) and $r_M = 0.25$ ($m_M \simeq 330$ MeV). Using

$$\mathrm{err}(f_{p,v}) = f_{p,v}^{eq.(37)} - f_{p,v}^{eq.(36)} f_{p,v}^{eq.(36)}, \tag{38}$$

to measure the error due to chiral logs that were not included in the extrapolation to the physical pion mass, we obtain

$$\underline{\mathsf{r_M = 0.15}}\qquad \mathrm{err}(f_p(v\cdot p = 0.19\ \mathrm{GeV})) \simeq -2\%\ , \qquad \mathrm{err}(f_p(v\cdot p = 0.54\ \mathrm{GeV})) \simeq -5\%\ ,$$

$$\underline{\mathsf{r_M = 0.25}}\qquad \mathrm{err}(f_p(v\cdot p = 0.19\ \mathrm{GeV})) \simeq -7\%\ , \qquad \mathrm{err}(f_p(v\cdot p = 0.54\ \mathrm{GeV})) \simeq -15\% \tag{39}$$

The above analysis applied to the form factor f_v leads to even smaller uncertainties: $\mathrm{err}(f_v)_{r_M=0.15} < 3\%$ and $\mathrm{err}(f_v)_{r_M=0.25} < 6\%$, for both values of $(v\cdot p)$.

This exercise is made just to illustrate how one can proceed in order to get an estimate of the systematic uncertainties due to the chiral extrapolation. As we saw, the amount of estimated uncertainty is highly sensitive to the choice of the point r_M, which is why the outcome of this exercise should be considered only as a rough estimate.

It is important to stress again that had we used the quenched expressions (32) to guide the chiral extrapolation, the result would stay above the result of the linear extrapolation, precisely the opposite to what happens in the full theory (which we show in fig. 7).

7 Conclusions

In this paper we explored the approach which contains in its core the leading order in the heavy quark expansion and the ChPT, to derive the expressions for the heavy-to-light pseudoscalar semileptonic form factors. The expressions are worked out in both the standard and quenched ChPT. In the latter case we observe the familiar feature of the quenched calculations, namely the appearance of the divergent chiral logarithms (quenched chiral logs) which makes it harder to compare with the formulae obtained in the standard ChPT for $B \to \pi$ form factors: one should find a fiducial window in which the masses are not so light so that the quenched chiral logs do not dominate the QChPT expressions, yet small enough for the chiral expansion to be meaningful. Furthermore, for the numerical estimates a number of low energy constants must be fixed by using both the available experimental data and the results of quenched lattice simulations. Our numerical estimates in which we use the QChPT expressions are highly sensitive to the value of the parameter g' (coupling of the doublet of heavy–light mesons to a light η'-meson). Until the lattice computation of that parameter is made, we are not able to make firm quantitative statements. Even information about the sign of g' would be welcome. It turns out that for $g' = -g$ (which, on the basis of the large N_c expansion, is the limiting value), we get a less pessimistic scenario, i.e. the quenching errors on $F_{+,T}(q^2)$ are not large for small recoils. One can even find the combinations of $(v\cdot p)$ and the light meson mass m_P, such that the quenching errors vanish. We were also able to find the combinations of $(v\cdot p)$ and m_P such that the quenching errors do not depend on the value of g'. At these points and for small recoils the quenching errors are between 15 and 25%. We reiterate that the numerical estimates do depend on the specific choice of the low energy constants.

As for the form factor $F_0(q^2)$, we observe that the quenching errors, as inferred from the present approach, are systematically large and the quenched value is larger than the unquenched one. This conclusion remains as such regardless of the value of the coupling g'. Only at the point corresponding to zero recoil are those errors reasonably small. Away from that point, they are large.

The same observations apply also when the final meson is a kaon. In that case, by using the QChPT expressions, we were able to verify that the form factors obtained for the kaon consisting of degenerated and non-degenerated quarks are effectively indistinguishable, provided the value of the coupling is $g' = -g$. For $g' = +g$ also those uncertainties become large.

Finally, the formulae presented in this work may be useful in assessing the systematic uncertainty due to chiral extrapolations. We showed how to include the chiral logs to extrapolate from the region in which the pion is heavier than 400 MeV. As could have been anticipated, the estimated uncertainty of the chiral extrapolation depends on the mass from which the chiral logs are included in the extrapolation.

We also provided the quenched chiral log coefficient, which might be useful if the lattice calculations are performed with very light mesons so that one could attempt to subtract the spurious effect of the quenched chiral log.

Acknowledgements:

We thank T. Onogi and G. Lacagnina for communicating the details of their works to us, S. Fajfer, J. Flynn, V. Lubicz, G. Martinelli and S. Sharpe for their comments on the manuscript, and Suzy Vascotto for her help in polishing the text. The work of S.P. and J.Z. has been supported in part by the Ministry of Education, Science and Sport of the Republic of Slovenia.

Appendix A: Chiral loop integrals

In this appendix we list the dimensionally regularized integrals encountered in the course of calculation. For more details, see [51] and references therein.

$$
\begin{aligned}
i\mu^{\epsilon}\int\frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}}\frac{1}{q^2-m^2} &= \frac{1}{16\pi^2}I_1(m)\ ,\\
i\mu^{\epsilon}\int\frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}}\frac{1}{(q^2-m^2)(q\cdot v-\Delta)} &= \frac{1}{16\pi^2}\frac{1}{\Delta}I_2(m,\Delta)\ ,
\end{aligned}
\tag{40}
$$

where

$$
\begin{aligned}
I_1(m) &= m^2\ln\Big(\frac{m^2}{\mu^2}\Big)-m^2\bar{\Delta}\ ,\\
I_2(m,\Delta) &= -2\Delta^2\ln\Big(\frac{m^2}{\mu^2}\Big)-4\Delta^2F\Big(\frac{m}{\Delta}\Big)+2\Delta^2(1+\bar{\Delta})\ ,
\end{aligned}
\tag{41}
$$

where $\bar{\Delta}=2/\epsilon-\gamma+\ln(4\pi)+1$. The function $F(x)$ has been calculated in ref. [52], for both the negative and positive values of the argument:

$$
F\left(\frac{1}{x}\right)=\begin{cases}-\dfrac{\sqrt{1-x^2}}{x}\left[\dfrac{\pi}{2}-\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)\right] & |x|\le 1\\[2ex] \dfrac{\sqrt{x^2-1}}{x}\ln\left(x+\sqrt{x^2-1}\right) & |x|\ge 1\ .\end{cases}
\tag{42}
$$

In addition to the integrals (40), one also needs the following two:

$$
\begin{aligned}
i\mu^{\epsilon}\int\frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}}\frac{q^\mu}{(q^2-m^2)(q\cdot v-\Delta)} &= \frac{v^\mu}{16\pi^2}\left[I_2(m,\Delta)+I_1(m)\right]\ ,\\
i\mu^{\epsilon}\int\frac{d^{4-\epsilon}q}{(2\pi)^{4-\epsilon}}\frac{q^\mu q^\nu}{(q^2-m^2)(q\cdot v-\Delta)} &= \frac{1}{16\pi^2}\Delta\left[J_1(m,\Delta)\eta^{\mu\nu}+J_2(m,\Delta)v^\mu v^\nu\right]\ ,
\end{aligned}
\tag{43}
$$

with

$$
\begin{aligned}
J_1(m,\Delta) &= (-m^2+\frac{2}{3}\Delta^2)\ln\left(\frac{m^2}{\mu^2}\right)+\frac{4}{3}(\Delta^2-m^2)F\left(\frac{m}{\Delta}\right)\\
&\qquad -\frac{2}{3}\Delta^2(1+\bar{\Delta})+\frac{1}{3}m^2(2+3\bar{\Delta})+\frac{2}{3}m^2-\frac{4}{9}\Delta^2\ ,
\end{aligned}
\tag{44a}
$$

$$
\begin{aligned}
J_2(m,\Delta) &= (2m^2-\frac{8}{3}\Delta^2)\ln\left(\frac{m^2}{\mu^2}\right)-\frac{4}{3}(4\Delta^2-m^2)F\left(\frac{m}{\Delta}\right)\\
&\qquad +\frac{8}{3}\Delta^2(1+\bar{\Delta})-\frac{2}{3}m^2(1+3\bar{\Delta})-\frac{2}{3}m^2+\frac{4}{9}\Delta^2\ .
\end{aligned}
\tag{44b}
$$

The functions $J_1(m,\Delta)$, $J_2(m,\Delta)$ differ from the ones in ref. [22] by the last two terms in (44) which are of $\mathcal{O}(m^2,\Delta^2)$. These additional (finite) terms originate from the fact that $\eta^{\mu\nu}$ is the $(4-\epsilon)$-dimensional metric tensor.

Appendix B: Explicit expressions for the one loop chiral corrections

As already mentioned in the body of the paper, the chiral loop corrections to the form factors can be written in the form

$$\delta f_{p,v} = \sum_I \delta f_{p,v}^{(I)} + \frac{1}{2}\delta Z_B^{\rm loop} + \frac{1}{2}\delta Z_P^{\rm loop}\,, \tag{45}$$

where the sum goes over all the graphs depicted in fig. 2. In what follows we give the explicit expressions for both form factors graph by graph and in both chiral theories (quenched and standard).

B.1 Quenched theory

In the calculation of one loop contributions, we made several approximations in order to simplify the final expressions. We make use of the fact that $v{\cdot}p > \Delta^*$ for the $B \to P$ transition and thus consistently neglect the mass differences between B, B^*, B_s and B_s^* mesons in the loops. This neglect induces a spurious singularity in the expression for the diagrams (7a) and (7b), at $v{\cdot}p \to 0$. To handle those singularities we follow the proposal by Falk and Grinstein [24] and resum the corresponding diagrams and then subtract the term that would renormalize the B^*-meson mass. We recall that a (b) superscripts distinguish the diagrams without (with) hairpin insertion. The formulae for the $B_j \to P_{ij}$ transition (with the valence quark content of mesons $B_j \sim b\bar{q}_j$ and $P_{ij} \sim q_i\bar{q}_j$) are expressed in terms of the pseudoscalar meson mass $M_i^2 = 4\mu_0 m_i$:

$$\begin{aligned}
\delta f_p^{(7a)} &= 6gg'(4\pi f)^2\ \left[J_1(M_i, v{\cdot}p) - \frac{1}{v{\cdot}p}\frac{2\pi}{3}M_i^3\right]\ ,\\
\delta f_p^{(7b)} &= -g^2(4\pi f)^2\left[\alpha_0 + \left(\alpha_0 M_i^2 - m_0^2\right)\frac{\partial}{\partial M_i^2}\right]\left(J_1(M_i, v{\cdot}p) - \frac{1}{v{\cdot}p}\frac{2\pi}{3}M_i^3\right)\,,\\
\delta f_p^{(9a)} &= gg'(4\pi f)^2\left[J_1(M_i, v{\cdot}p) + J_1(M_j, v{\cdot}p) - \frac{1}{v{\cdot}p}\frac{2\pi}{3}\left(M_i^3 + M_j^3\right)\right]\,,\\
\delta f_p^{(9b)} &= g^2 3(4\pi f)^2\Bigg[\alpha_0 M_j^2 - m_0^2 M_j^2 - M_i^2\left[\frac{1}{v{\cdot}p}\frac{2\pi}{3}M_j^3 - J_1(M_j, v{\cdot}p)\right]\\
&\qquad -\alpha_0 M_i^2 - m_0^2 M_j^2 - M_i^2\left[\frac{1}{v{\cdot}p}\frac{2\pi}{3}M_i^3 - J_1(M_i, v{\cdot}p)\right]\Bigg]\,,\\
\delta f_p^{(12b)} &= \frac{T}{18(4\pi f)^2}\Bigg\{\frac{2m_0^2 - \alpha_0(M_i^2 + M_j^2)}{M_j^2 - M_i^2}\left(I_1(M_j) - I_1(M_i)\right)\\
&\qquad + \left(\alpha_0 M_i^2 - m_0^2\right)\frac{\partial I_1(M_i)}{\partial M_i^2} + \left(\alpha_0 M_j^2 - m_0^2\right)\frac{\partial I_1(M_j)}{\partial M_j^2}\Bigg\}\,,\\
\delta f_p^{(13a)} &= -\frac{iV_L'(0)f}{\sqrt{6}(4\pi f)^2}I_1(M_i)\ ,
\end{aligned}$$

$$\delta f_p^{(13b)} = \frac{1}{6(4\pi f)^2}\Big[\alpha_0 I_1(M_i) + \Big(\alpha_0 M_i^2 - m_0^2\Big)\frac{\partial I_1(M_i)}{\partial M_i^2}\Big]\,, \tag{46}$$

where $T = 0$ for $i = j$, and $T = 1$ otherwise. The functions $I_1(m)$, $J_1(m,\Delta)$ are given in appendix A. As for the form factor $f_v(v\cdot p)$, the non-zero chiral loop corrections are

$$\begin{aligned}
\delta f_v^{(4a)} &= -\frac{iTV_L'(0)f}{\sqrt{6}(4\pi f)^2}\Big[I_2(M_j, v\cdot p) - I_2(M_i, v\cdot p) + 12\left(I_1(M_j) - I_1(M_i)\right)\Big] \\
\delta f_v^{(4b)} &= \frac{T}{6(4\pi f)^2}\Bigg\{\frac{1}{M_j^2 - M_i^2}\Big[\left(\alpha_0 M_j^2 - m_0^2\right)\left[I_1(M_j) + 2I_2(M_j, v\cdot p)\right] \\
&\qquad - \left(\alpha_0 M_i^2 - m_0^2\right)\left(I_1(M_i) + 2I_2(M_i, v\cdot p)\right)\Big] \\
&\qquad - \left(\alpha_0 + (\alpha_0 M_i^2 - m_0^2)\frac{\partial}{\partial M_i^2}\right)\left(I_1(M_i) + 2I_2(M_i, v\cdot p)\right)\Bigg\} \\
\delta f_v^{(14a)} &= -\frac{iV_L'(0)f}{2\sqrt{6}(4\pi f)^2}\left(I_1(M_i) + I_1(M_j)\right) \\
\delta f_v^{(14b)} &= \frac{1}{18(4\pi f)^2}\Bigg\{(\alpha_0 M_j^2 - m_0^2)\Big[\frac{I_1(M_j)}{M_j^2 - M_i^2} + \frac{\partial I_1(M_j)}{\partial M_j^2}\Big] \\
&\qquad + (\alpha_0 M_i^2 - m_0^2)\Big[\frac{I_1(M_i)}{M_i^2 - M_j^2} + \frac{\partial I_1(M_i)}{\partial M_i^2}\Big] + \alpha_0\left(I_1(M_i) + I_1(M_j)\right)\Bigg\}
\end{aligned} \tag{47}$$

In the wave function renormalization factors, $Z_{B,P}$, we separate the one-loop chiral contribution $\delta Z_{B,P}^{\text{loop}}$ from the pieces coming from the counter-terms $\delta Z_{B,P}^{c.t.}$, i.e.

$$Z_{B,P} = 1 + \delta Z_{B,P} = 1 + \delta Z_{B,P}^{\text{loop}} + \delta Z_{B,P}^{c.t.}\ . \tag{48}$$

More specifically

$$\begin{aligned}
\delta Z_{B_j}^{\text{loop}} &= 1(4\pi f)^2\Big[(2g^2\alpha_0 M_j^2 - 6gg' M_j^2 - g^2 m_0^2)\ln\Big(\frac{M_j^2}{\mu^2}\Big) \\
&\qquad + \alpha_0 g^2 M_j^2 - m_0^2 g^2 + \Big(-2g^2 M_j^2\alpha_0 + 6gg' M_j^2 + g^2 m_0^2\Big)\bar{\Delta}\Big]
\end{aligned} \tag{49a}$$

$$\begin{aligned}
\delta Z_{P_{ij}}^{\text{loop}} &= 19(4\pi f)^2\Big\{\frac{\ln\left(M_j^2/M_i^2\right)}{M_j^2 - M_i^2}\left(2\alpha_0 M_j^2 M_i^2 - m_0^2(M_j^2 + M_i^2)\right) \\
&\qquad + 2m_0^2 - \alpha_0(M_i^2 + M_j^2)\Big\}\ ,
\end{aligned} \tag{49b}$$

while the counter-terms contribute as

$$\delta Z_{B_j}^{c.t.} = k_1 m_j\ , \qquad\qquad \delta Z_{P_{ij}}^{c.t.} = -8L_5\frac{4\mu_0}{f^2}(m_i + m_j)\ . \tag{50}$$

Thus the wave function renormalization factor for π and K mesons reads

$$Z_\pi = 1 - 8L_5 \frac{4\mu_0}{f^2}\ 2\hat{m} \tag{51a}$$

$$Z_K = 1 - \frac{1}{9(4\pi f)^2}\left\{ \frac{\ln\left(m_s/\hat{m}\right)}{(m_K^2 - m_\pi^2)}\left(\alpha_0 m_\pi^4 - \alpha_0 2m_K^2 m_\pi^2 + m_0^2 m_K^2\right)\right.$$
$$\left. + 2\alpha_0 m_K^2 - 2m_0^2 \right\} - 8L_5 \frac{4\mu_0}{f^2}(\hat{m} + m_s)\ , \tag{51b}$$

where we ignore the isospin-breaking effects and set $m_u = m_d = \hat{m}$.

Finally, we also display the expressions for the heavy–light meson decay constant

$$f_{B_i} = \frac{\alpha}{\sqrt{m_B}}\left\{1 + \frac{1}{6(4\pi f)^2}\left[\left(\alpha_0 - if\sqrt{6}V_L'(0)\right) I_1(M_i) + (\alpha_0 M_i^2 - m_0^2)\frac{\partial I_1(M_i)}{\partial M_i^2}\right]\right.$$
$$\left. + \varkappa_1 m_i + \tfrac{1}{2}\delta Z_{B_i}\right\}, \tag{52}$$

in agreement with refs. [15, 16].

B.2 Full (unquenched) theory

In this subsection we present the expressions for the form factors in the full theory. The non-analytic contributions to form factors in this theory have already been calculated in ref. [24]. Our results also include the analytic terms. As in the quenched case, we work in the isospin limit $m_u = m_d = \hat{m}$ and neglect the differences of heavy meson masses in the loops. We now list the results for $\delta f_{v,p}^{(I)}$ in $B_j \to P_{ij}$ mediated by the $(V - A)$ operator, where P_{ij} stands for the light pseudoscalar meson with the valence quark content $\bar{q}_i q_j$.

The form factor $f_p(v{\cdot}p)$ receives the following 1-loop corrections:

$$\delta f_p^{(7a)} = \frac{3g^2}{(4\pi f)^2}\left\{\sum_{P'} C_{B_jP'P_{ij}}^{(7a)}\left[J_1(m_{P'}, v{\cdot}p) - \frac{1}{v{\cdot}p}\frac{2\pi}{3}m_{P'}^3\right]\right\},$$

$$\delta f_p^{(9a)} = -\frac{g^2}{(4\pi f)^2}\left\{\sum_{P'} C_{B_jP'P_{ij}}^{(9a)}\left[J_1(m_{P'}, v{\cdot}p) - \frac{1}{v{\cdot}p}\frac{2\pi}{3}m_{P'}^3\right]\right\},$$

$$\delta f_p^{(12a)} = \frac{1}{(4\pi f)^2}\left[\sum_{P'} C_{B_jP'P_{ij}}^{(12a)} I_1(m_{P'})\right],$$

$$\delta f_p^{(13a)} = \frac{1}{(4\pi f)^2}\left[\sum_{P'} C_{B_jP'P_{ij}}^{(13a)} I_1(m_{P'})\right], \tag{53}$$

where the coefficients $C_{B_jP'P_{ij}}$ depend on the final and initial states. A detailed list of coefficients includes

- FOR $B \to K$ TRANSITION:
  $C_{B\pi K}^{(7a)} = 0,\quad C_{BKK}^{(7a)} = 2,\quad C_{B\eta K}^{(7a)} = 2/3;\quad C_{B\pi K}^{(9a)} = 0,\quad C_{BKK}^{(9a)} = 0,\quad C_{B\eta K}^{(9a)} = \frac{1}{3};$
  $C_{B\pi K}^{(12a)} = -\frac{1}{4},\quad C_{BKK}^{(12a)} = -\frac{1}{2},\quad C_{B\eta K}^{(12a)} = -\frac{1}{4};\quad C_{B\pi K}^{(13a)} = 0,\quad C_{BKK}^{(13a)} = -1,\quad C_{B\eta K}^{(13a)} = -\frac{1}{3};$

- FOR $B \to \pi$ TRANSITION:
  $C^{(7a)}_{B\pi\pi} = \frac{3}{2}$, $C^{(7a)}_{BK\pi} = 1$, $C^{(7a)}_{B\eta\pi} = \frac{1}{6}$; $C^{(9a)}_{B\pi\pi} = \frac{1}{2}$, $C^{(9a)}_{BK\pi} = 0$, $C^{(9a)}_{B\eta\pi} = -\frac{1}{6}$;
  $C^{(12a)}_{B\pi\pi} = -\frac{2}{3}$, $C^{(12a)}_{BK\pi} = -\frac{1}{3}$, $C^{(12a)}_{B\eta\pi} = 0$; $C^{(13a)}_{B\pi\pi} = -\frac{3}{4}$, $C^{(13a)}_{BK\pi} = -\frac{1}{2}$, $C^{(13a)}_{B\eta\pi} = -\frac{1}{12}$;
- FOR $B_s \to K$ TRANSITION:
  $C^{(7a)}_{B_s\pi K} = \frac{3}{2}$, $C^{(7a)}_{B_sKK} = 1$, $C^{(7a)}_{B_s\eta K} = \frac{1}{6}$; $C^{(9a)}_{B_s\pi K} = 0$, $C^{(9a)}_{B_sKK} = 0$, $C^{(9a)}_{B_s\eta K} = \frac{1}{3}$;
  $C^{(12a)}_{B_s\pi K} = -\frac{1}{4}$, $C^{(12a)}_{B_sKK} = -\frac{1}{2}$, $C^{(12a)}_{B_s\eta K} = -\frac{1}{4}$; $C^{(13a)}_{B_s\pi K} = -\frac{3}{4}$, $C^{(13a)}_{B_sKK} = -\frac{1}{2}$, $C^{(13a)}_{B_s\eta K} = -\frac{1}{12}$;

The non-vanishing one-loop chiral corrections to $f_v(v\cdot p)$ form factor are

$$\delta f_v^{(4a)} = \frac{1}{(4\pi f)^2}\left\{\sum_{P'} D^{(4a)}_{B_jP'P_{ij}}\left[I_2(m_{P'}, v\cdot p) + \frac{1}{2}I_1(m_{P'})\right]\right\} ,$$

$$\delta f_v^{(14a)} = \frac{1}{(4\pi f)^2}\left[\sum_{P'} D^{(14a)}_{B_jP'P_{ij}} I_1(m_{P'})\right] , \tag{54}$$

where the process-dependent coefficients $D_{B_jP'P_{ij}}$ are

- FOR $B \to K$ TRANSITION:
  $D^{(4a)}_{B\pi K} = 0, D^{(4a)}_{BKK} = 2, D^{(4a)}_{B\eta K} = 1; D^{(14a)}_{B\pi K} = -1/4, D^{(14a)}_{BKK} = -1/2, D^{(14a)}_{B\eta K} = -1/12$;
- FOR $B \to \pi$ TRANSITION:
  $D^{(4a)}_{B\pi\pi} = 2, D^{(4a)}_{BK\pi} = 1, D^{(4a)}_{B\eta\pi} = 0; D^{(14a)}_{B\pi\pi} = -5/12, D^{(14a)}_{BK\pi} = -1/3, D^{(14a)}_{B\eta\pi} = -1/12$;
- FOR $B_s \to K$ TRANSITION:
  $D^{(4a)}_{B_s\pi K} = 3/2$, $D^{(4a)}_{B_sKK} = 1$, $D^{(4a)}_{B_s\eta K} = 1/2$; $D^{(14a)}_{B_s\pi K} = -1/4$, $D^{(14a)}_{B_sKK} = -1/2$, $D^{(14a)}_{B_s\eta K} = -1/12$;

The wave function renormalization factors Z for B mesons in the full theory are

$$Z_{B_{u,d}} = 1 - \frac{3g^2}{(4\pi f)^2}\left[\frac{3}{2}I_1(m_\pi) + I_1(m_K) + \frac{1}{6}I_1(m_\eta)\right] + k_1\hat{m} + k_2(m_u + m_d + m_s) ,$$

$$Z_{B_s} = 1 - \frac{3g^2}{(4\pi f)^2}\left[2I_1(m_K) + \frac{2}{3}I_1(m_\eta)\right] + k_1 m_s + k_2(m_u + m_d + m_s) , \tag{55}$$

while for the light mesons we have

$$Z_K = 1 + \frac{1}{(4\pi f)^2}\left[\frac{1}{2}I_1(m_\pi) + I_1(m_K) + \frac{1}{2}I_1(m_\eta)\right] - 8L_5\frac{4\mu_0}{f^2}(\hat{m} + m_s) - 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s) ,$$

$$Z_\pi = 1 + \frac{2}{3(4\pi f)^2}\left[2I_1(m_\pi) + I_1(m_K)\right] - 8L_5\frac{4\mu_0}{f^2}\, 2\hat{m} - 16L_4\frac{4\mu_0}{f^2}(m_u + m_d + m_s) . \tag{56}$$

As in the previous subsection we close the list of results by showing also the corresponding formulae for the heavy–light decay constants. We have

$$
\begin{aligned}
f_{B_s} &= \frac{\alpha}{\sqrt{m_B}}\Big(1 - \frac{1}{(4\pi f)^2}\big[I_1(m_K) + \frac{1}{3}I_1(m_\eta)\big] + \varkappa_1 m_s + \varkappa_2(m_u + m_d + m_s) + \frac{1}{2}\delta Z_{B_s}\Big)\ , \\
f_{B_{u,d}} &= \frac{\alpha}{\sqrt{m_B}}\Big(1 - \frac{1}{(4\pi f)^2}\big[\frac{3}{4}I_1(m_\pi) + \frac{1}{2}I_1(m_K) + \frac{1}{12}I_1(m_\eta)\big] \\
&\qquad + \varkappa_1 \hat{m} + \varkappa_2(m_u + m_d + m_s) + \frac{1}{2}\delta Z_{B_{u,d}}\Big)\ ,
\end{aligned}
\tag{57}
$$

in agreement with results of refs. [15, 16].

References

[1] N. Isgur and M. B. Wise, Phys. Rev. D **42** (1990) 2388.

[2] J. Charles, A. Le Yaouanc, L. Oliver, O. Pène and J. C. Raynal, Phys. Rev. D **60** (1999) 014001 [hep-ph/9812358].

[3] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, Phys. Rev. D **63** (2001) 114020 [hep-ph/0011336].

[4] P. Colangelo and A. Khodjamirian, *"QCD sum rules: A modern perspective"*, hep-ph/0010175; V. M. Braun, *"QCD sum rules for heavy flavors"*, hep-ph/9911206.

[5] K. C. Bowler *et al.* [UKQCD Collaboration], Phys. Lett. B **486** (2000) 111 [hep-lat/9911011].

[6] A. Abada, D. Becirevic, P. Boucaud, J. P. Leroy, V. Lubicz and F. Mescia, Nucl. Phys. B **619** (2001) 565 [hep-lat/0011065].

[7] S. Aoki *et al.* [JLQCD Collaboration], Phys. Rev. D **64** (2001) 114505 [hep-lat/0106024].

[8] J. Shigemitsu *et al., "Semileptonic B Decays from an NRQCD/D234 Action"*, hep-lat/0208062.

[9] A. X. El-Khadra, A. S. Kronfeld, P. B. Mackenzie, S. M. Ryan and J. N. Simone, Phys. Rev. D **64** (2001) 014502 [hep-ph/0101023].

[10] S. R. Sharpe, Phys. Rev. D **46** (1992) 3146 [hep-lat/9205020].

[11] C. W. Bernard and M. F. Golterman, Phys. Rev. D **46** (1992) 853 [hep-lat/9204007].

[12] P. H. Damgaard and K. Splittorff, Phys. Rev. D **62** (2000) 054509 [hep-lat/0003017].

[13] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).

[14] H. Leutwyler, *"Chiral dynamics"*, hep-ph/0008124; A. Pich, Rep. Prog. Phys. **58** (1995) 563 [hep-ph/9502366]; G. Ecker, *"Strong interactions of light flavours"*, hep-ph/0011026; U. G. Meissner, Rep. Prog. Phys. **56** (1993) 903 [hep-ph/9302247]; J. F. Donoghue, E. Golowich and B. R. Holstein, *"Dynamics Of The Standard Model"*, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. **2** (1992) 1.

[15] M. J. Booth, Phys. Rev. D **51** (1995) 2338 [hep-ph/9411433].

[16] S. R. Sharpe and Y. Zhang, Phys. Rev. D **53** (1996) 5125 [hep-lat/9510037].

[17] M. J. Savage, Phys. Rev. D **65**, 034014 (2002) [hep-ph/0109190].

[18] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gatto, F. Feruglio and G. Nardulli, Phys. Rep. **281** (1997) 145 [hep-ph/9605342]; B. Grinstein, hep-ph/9508227; A. V. Manohar and M. B. Wise, *"Heavy Quark Physics"*, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. **10** (2000) 1.

[19] G. Burdman and J. F. Donoghue, Phys. Lett. B **280** (1992) 287; M. B. Wise, Phys. Rev. D **45** (1992) 2188; J. L. Goity, Phys. Rev. D **46**, 3929 (1992), hep-ph/9206230.

[20] For an introduction to HQET see e.g. M. Neubert, Phys. Rep. **245**, 259 (1994) [hep-ph/9306320].

[21] E. Eichten and B. Hill, Phys. Lett. B **234** (1990) 511; D. J. Broadhurst and A. G. Grozin, Phys. Rev. D **52** (1995) 4082 [hep-ph/9410240].

[22] C. G. Boyd and B. Grinstein, Nucl. Phys. B **442** (1995) 205 [hep-ph/9402340].

[23] G. Burdman, Z. Ligeti, M. Neubert and Y. Nir, Phys. Rev. D **49** (1994) 2331 [hep-ph/9309272].

[24] A. F. Falk and B. Grinstein, Nucl. Phys. B **416** (1994) 771 [hep-ph/9306310].

[25] M. F. Golterman, Acta Phys. Polon. B **25** (1994) 1731 [hep-lat/9411005].

[26] S. R. Sharpe, Nucl. Phys. Proc. Suppl. **53** (1997) 181 [hep-lat/9609029].

[27] C. Bernard *et al.* [MILC Collaboration], *"Lattice calculation of heavy–light decay constants with two flavors of dynamical quarks"*, hep-lat/0206016.

[28] A. Ali Khan *et al.* [CP-PACS Collaboration], Phys. Rev. D **64** (2001) 034505 [hep-lat/0010009].

[29] K. C. Bowler *et al.* [UKQCD Collaboration], Nucl. Phys. B **619** (2001) 507 [hep-lat/0007020].

[30] D. Becirevic, P. Boucaud, J. P. Leroy, V. Lubicz, G. Martinelli, F. Mescia and F. Rapuano, Phys. Rev. D **60** (1999) 074501 [hep-lat/9811003].

[31] S. Narison, *"Light and heavy quark masses, flavor breaking of chiral condensates, meson weak leptonic decay constants in QCD"*, hep-ph/0202200.

[32] A. A. Penin and M. Steinhauser, Phys. Rev. D **65** (2002) 054006 [hep-ph/0108110].

[33] S. Ahmed *et al.* [CLEO Collaboration], Phys. Rev. Lett. **87** (2001) 251801 [hep-ex/0108013];
A. Anastassov *et al.* [CLEO Collaboration], Phys. Rev. D **65** (2002) 032003 [hep-ex/0108043].

[34] A. Abada *et al.*, *"First lattice QCD estimate of the $g_{D^*D\pi}$ coupling,"* hep-ph/0206237, hep-lat/0209092.

[35] A. Khodjamirian, R. Rückl, S. Weinzierl and O. I. Yakovlev, Phys. Lett. B **457** (1999) 245 [hep-ph/9903421].

[36] G. M. de Divitiis, L. Del Debbio, M. Di Pierro, J. M. Flynn, C. Michael and J. Peisa [UKQCD Collaboration], JHEP **9810** (1998) 010 [hep-lat/9807032].

[37] H. Leutwyler, Phys. Lett. B **378** (1996) 313 [hep-ph/9602366].

[38] S. Aoki *et al.* [CP-PACS Collaboration], *"Light hadron spectrum and quark masses from quenched lattice QCD"*, hep-lat/0206009.

[39] S. Aoki *et al.* [JLQCD Collaboration], Phys. Rev. D **62** (2000) 094501 [hep-lat/9912007].

[40] T. Blum *et al.*, *"Quenched lattice QCD with domain wall fermions and the chiral limit"*, hep-lat/0007038.

[41] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. **175**, 2195 (1968).

[42] M. Jamin, Phys. Lett. B **538** (2002) 71 [hep-ph/0201174].

[43] D. Becirevic, V. Lubicz and C. Tarantino [SPQ(cd)R Collaboration], *"Continuum determination of light quark masses from quenched lattice QCD"*, hep-lat/0208003.

[44] J. Heitger, R. Sommer and H. Wittig [ALPHA Collaboration], Nucl. Phys. B **588** (2000) 377 [hep-lat/0006026].

[45] G. Colangelo, S. Dürr and R. Sommer, *"Finite size effects on M_π in QCD from Chiral Perturbation Theory"*, hep-lat/0209110.

[46] W. Bardeen, A. Duncan, E. Eichten and H. Thacker, Phys. Rev. D **62** (2000) 114505 [hep-lat/0007010].

[47] T. Blum *et al.* [RBC Collaboration], *"Kaon matrix elements and CP-violation from quenched lattice QCD. I: The 3-flavor case"*, hep-lat/0110075 (see around page 37).

[48] T. DeGrand and U. M. Heller [MILC collaboration], Phys. Rev. D **65** (2002) 114501 [hep-lat/0202001].

[49] I. Horvath *et al.*, Phys. Rev. D **66** (2002) 034501 [hep-lat/0201008].

[50] H. Wittig, *"Chiral Effective Lagrangian and Quark Masses"*, hep-lat/0209162.

[51] J. Zupan, Eur. Phys. J. C **25** (2002) 233 [hep-ph/0202135]; A. O. Bouzas, Eur. Phys. J. C **12** (2000) 643 [hep-ph/9910536].

[52] I. W. Stewart, Nucl. Phys. B **529** (1998) 62 [hep-ph/9803227].